Permutation patterns and permutation groups

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Outline

Permutation patterns

The "Galois" connection $\mathrm{Pat}^{(\ell)} - \mathrm{Comp}^{(n)}$

Questions and answers

Further problems

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Further problems

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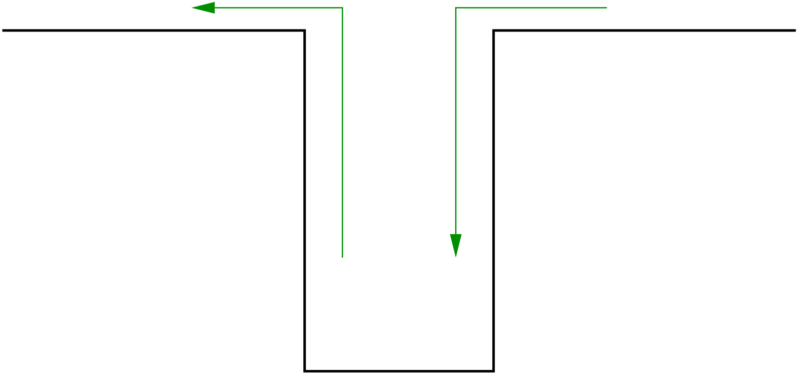

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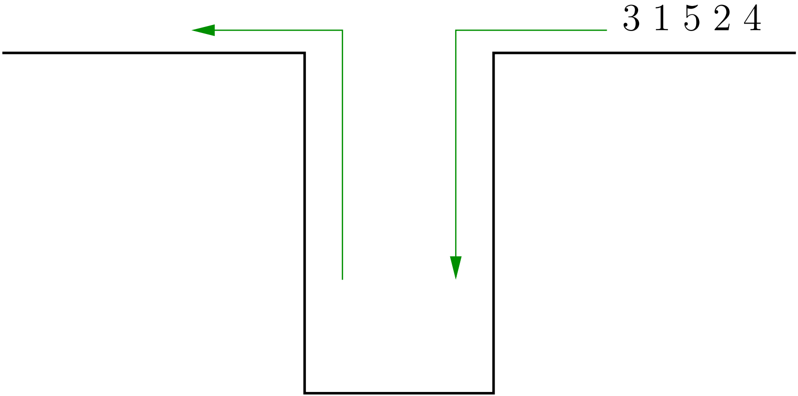

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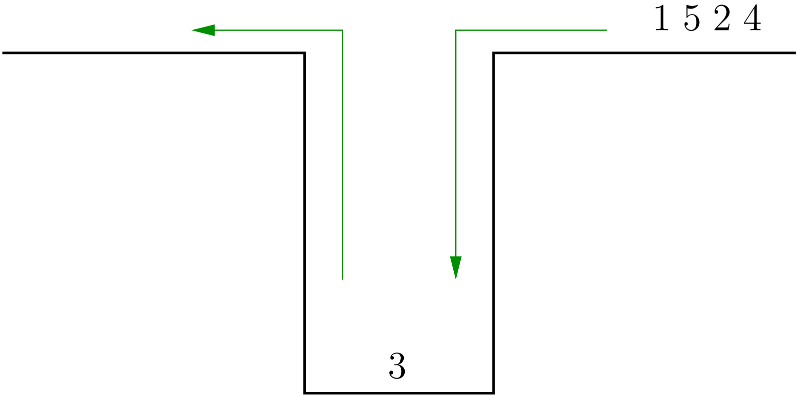

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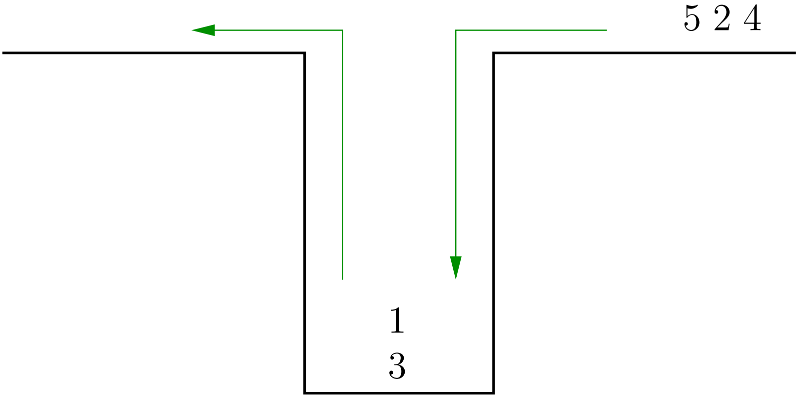

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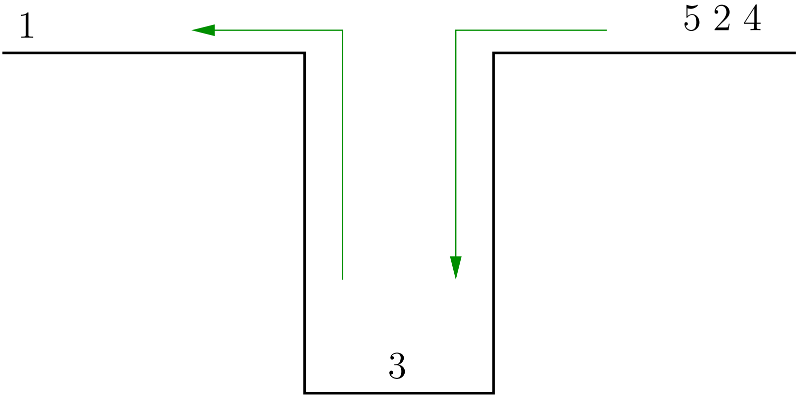

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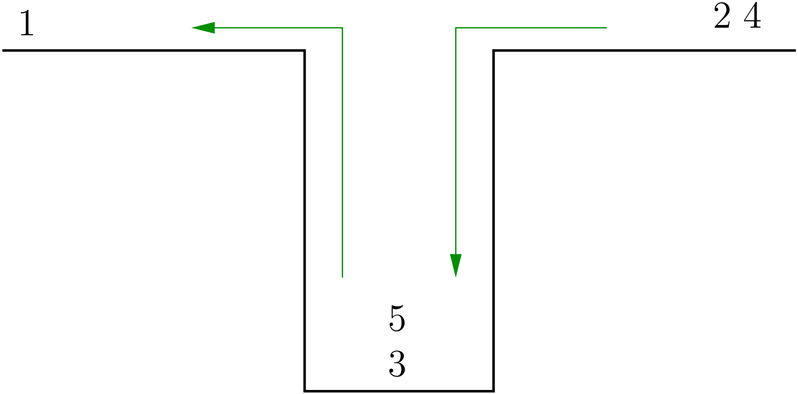

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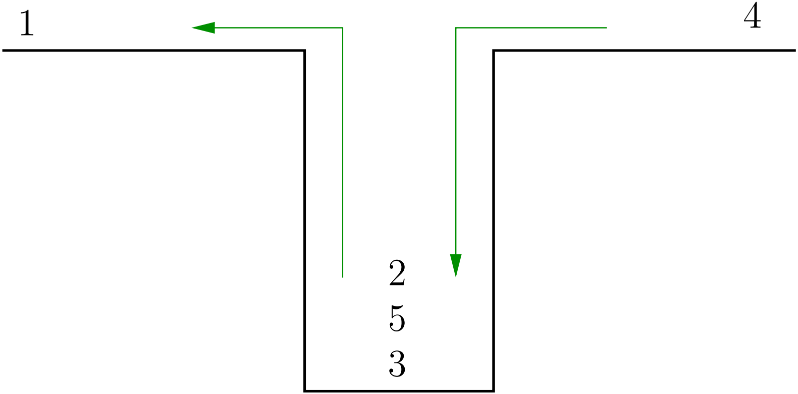

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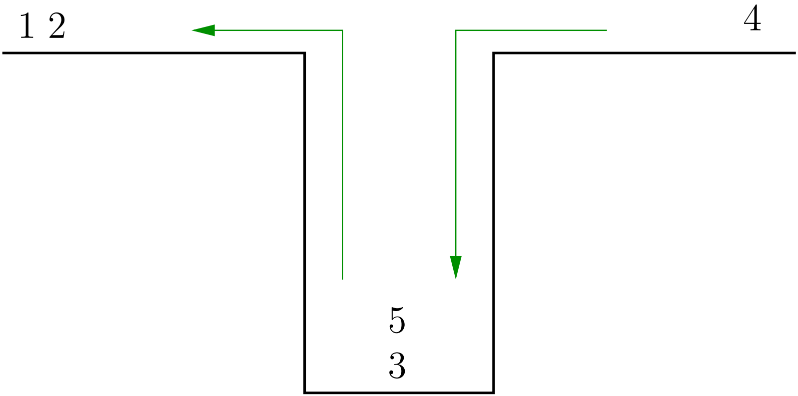

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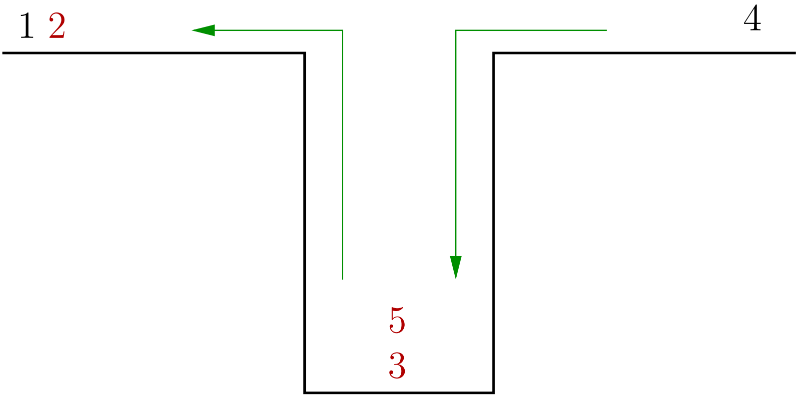

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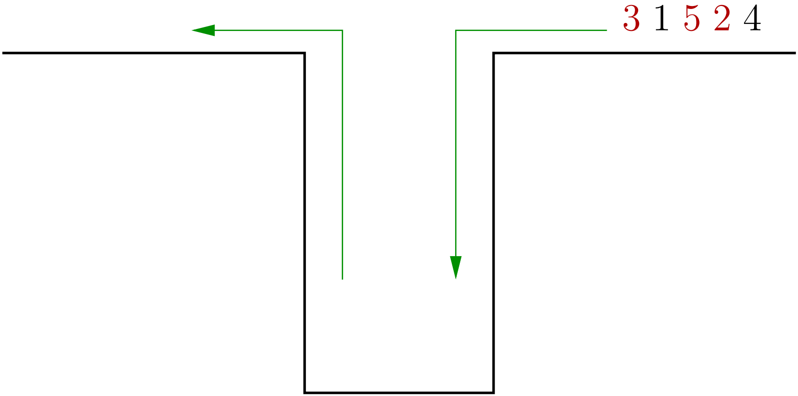

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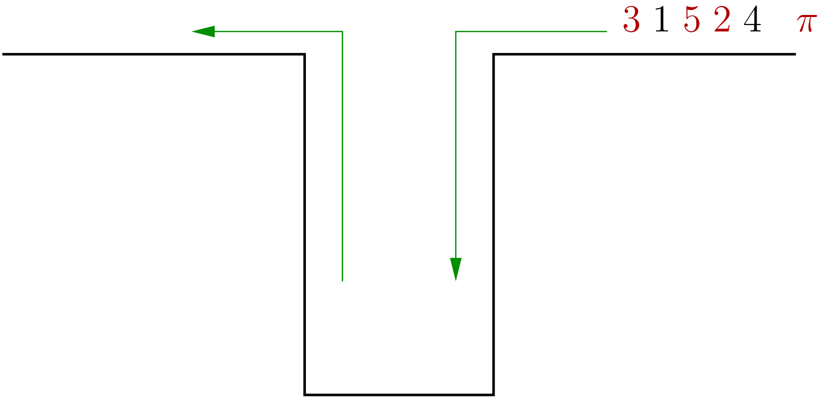

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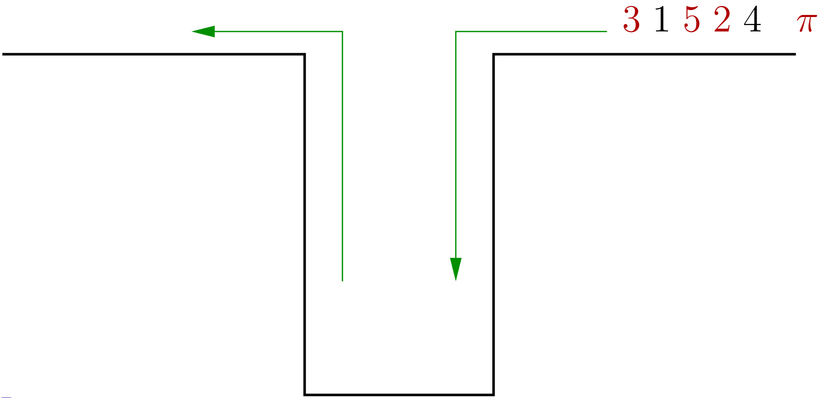

Proposition

A permutation π can be sorted by a stack if and only if it does not contain a subsequence $\ldots a \ldots b \ldots c \ldots$ with $c < a < b$.

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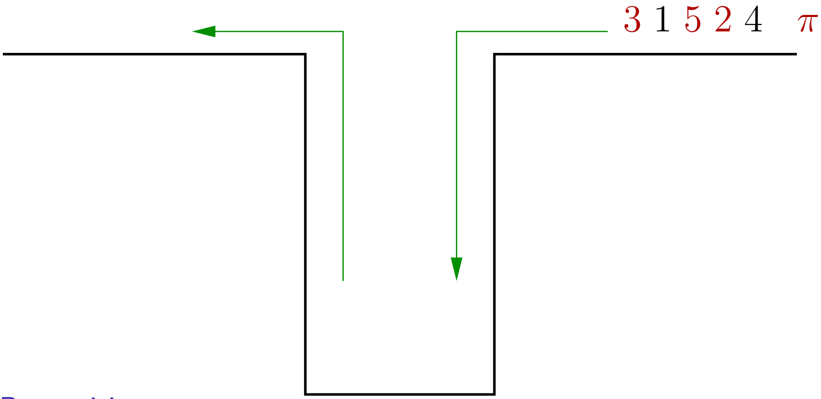

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i.e., such *"patterns" abc* (like 352) must be avoided

Permutations

A permutation $\pi \in S_n$ (bijection $\pi \in [n]^{[n]}$, $[n] := \{1, \ldots, n\}$) will be considered as a word (n-tuple $\pi \in [n]^n$) of length n:

$$(\pi_1, \ldots, \pi_n) := (\pi(1), \ldots, \pi(n)).$$

e.g. $\pi = (31524) \in [5]^5$ is the permutation
$1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 5$, $4 \mapsto 2$, $5 \mapsto 4$

graphical representation:

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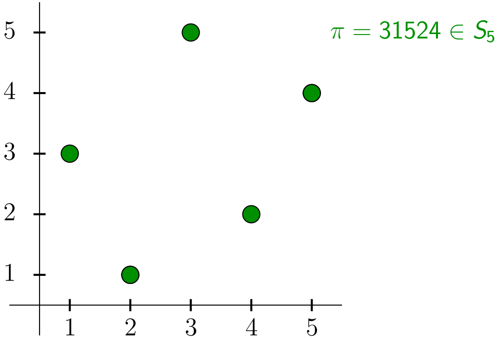
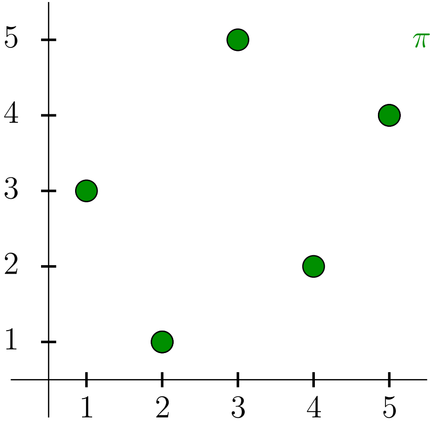


Permutation patterns (Example)



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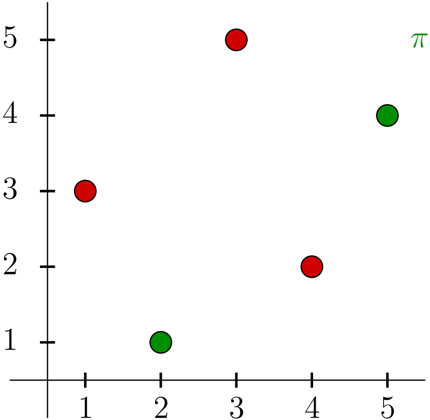

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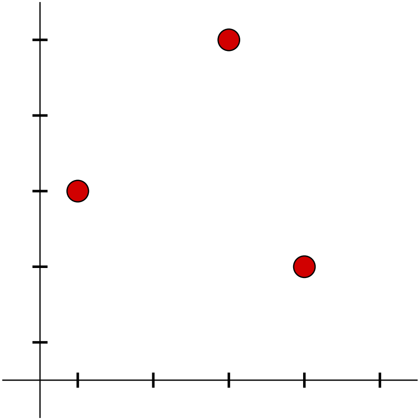

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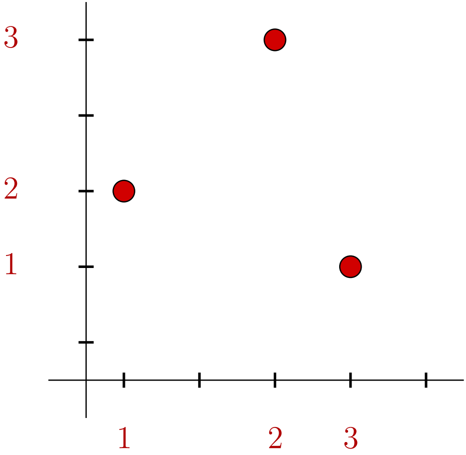

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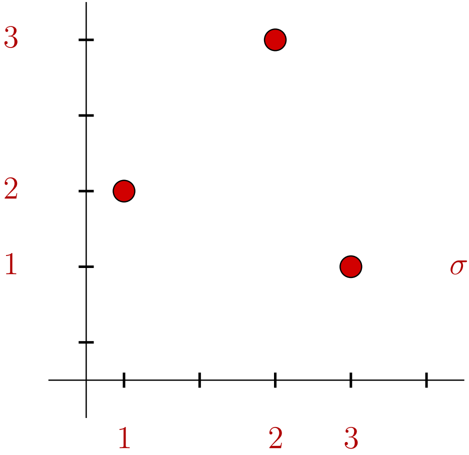

$\sigma = 231 \in S_3$

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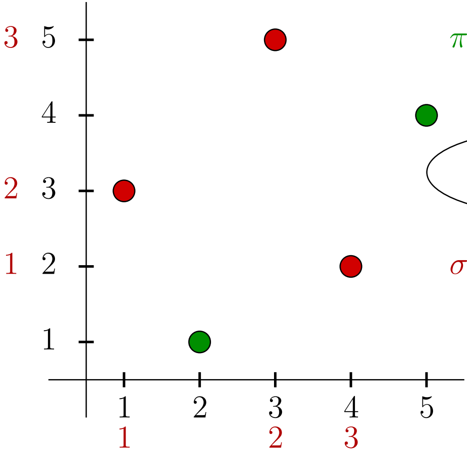

$\pi = 31524 \in S_5$

π involves σ

$\sigma \leq \pi$

σ is pattern of π

$\sigma = 231 \in S_3$

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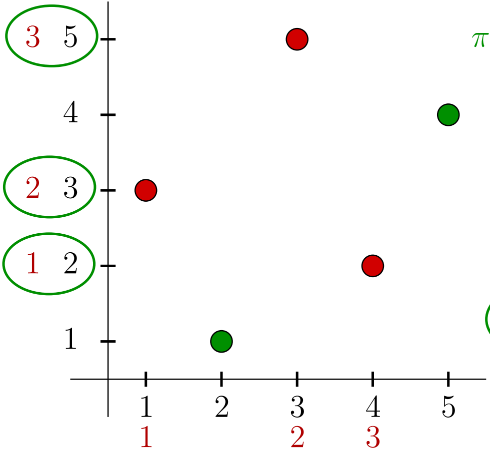

$\pi = 31524 \in S_5$

352 substring

231 reduced form

$231 = \mathrm{red}(352)$

Permutation patterns

$\ell \leq n$, $\sigma \in S_\ell$, $\pi \in S_n$

$\sigma \leq \pi :\iff$ there exists a substring $\mathbf{u}$ of π of length ℓ such that $\sigma = \mathrm{red}(\mathbf{u})$

(σ is *ℓ-pattern* of π, or π *involves* σ)

π *avoids* $\sigma :\iff \sigma \not\leq \pi$.

$\mathrm{Pat}^{(\ell)}\,\pi := \{\sigma \in S_\ell \mid \sigma \leq \pi\}$

The pattern involvement relation $\leq$ is a partial order on the set $\mathbb{P} := \bigcup_{n \geq 1} S_n$ of all finite permutations and we have:

$\ell \leq m \leq n$, $\sigma \in S_\ell$, $\pi \in S_m$, $\tau \in S_n$:

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Closed classes of permutations

M.D. Atkinson and R. Beals ([1999], *Permuting mechanisms and closed classes of permutations*) consider *closed classes of permutations*, i.e., subsets $A \subseteq \mathbb{P} = \bigcup_{n \geq 1} S_n$ which are closed under taking patterns (order ideal in $(\mathbb{P}, \leq)$):

$$\pi \in A,\ \sigma \leq \pi \implies \sigma \in A.$$

Then, for the sequence

$$(A_1, A_2, \ldots, A_\ell, \ldots, A_n, \ldots)$$

where $A_\ell := A \cap S_\ell$, we get $\mathrm{Pat}^{(\ell)} A_n \subseteq A_\ell$ for $\ell \leq n$.

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A (monotone) Galois connection

The relation $\sigma \not\leq \pi$ (π *avoids* σ) induces a Galois connection between subsets of S_ℓ and S_n. The corresponding "monotone Galois connection" (residuation) is given by the following operators (monotone w.r.t. $\subseteq$):

For $S \subseteq S_\ell$, $T \subseteq S_n$ ($\ell \leq n$) let

$$\mathrm{Comp}^{(n)} S := \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)} \tau \subseteq S\} = \{\tau \in S_n \mid \forall \sigma' \in S_\ell \setminus S : \sigma' \not\leq \tau\},$$

$$\mathrm{Pat}^{(\ell)} T := \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)} \tau = S_\ell \setminus \{\sigma' \in S_\ell \mid \forall \tau \in T : \sigma' \not\leq \tau\}.$$

Then we have:

$$\mathrm{Pat}^{(\ell)} \mathrm{Comp}^{(n)} S \subseteq S \quad \text{(kernel operator)},$$

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For $S \subseteq S_\ell$ there is an "increasing" sequence of (Galois) closures $\mathrm{Comp}^{(n)} S \subseteq S_n$:

$$S, \mathrm{Comp}^{(\ell+1)} S, \dots, \mathrm{Comp}^{(n)} S, \mathrm{Comp}^{(n+1)} S, \dots$$

Observation: For $\ell \leq m \leq n$ and $S \subseteq S_\ell$, $T \subseteq S_n$ we have $\mathrm{Comp}^{(n)} \mathrm{Comp}^{(m)} S = \mathrm{Comp}^{(n)} S$ (and $\mathrm{Pat}^{(\ell)} \mathrm{Pat}^{(m)} T = \mathrm{Pat}^{(\ell)} T$).

For a closed class of permutations $(A_1, \dots, A_\ell, \dots, A_n, \dots)$ we have $\mathrm{Pat}^{(\ell)} A_n \subseteq A_\ell$ and thus $A_n \subseteq \mathrm{Comp}^{(n)} A_\ell$ for $\ell \leq n$.

Example:

$(\mathrm{Pat}^{(1)} S, \dots, \mathrm{Pat}^{(k)} S, \dots, \mathrm{Pat}^{(\ell-1)} S, S, \mathrm{Comp}^{(\ell+1)} S, \dots, \mathrm{Comp}^{(n)} S, \dots)$

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For $S \subseteq S_\ell$ there is an "increasing" sequence of (Galois) closures $\mathrm{Comp}^{(n)} S \subseteq S_n$:

$$S, \mathrm{Comp}^{(\ell+1)} S, \ldots, \mathrm{Comp}^{(n)} S, \mathrm{Comp}^{(n+1)} S, \ldots$$

Observation: For $\ell \leq m \leq n$ and $S \subseteq S_\ell$, $T \subseteq S_n$ we have $\mathrm{Comp}^{(n)} \mathrm{Comp}^{(m)} S = \mathrm{Comp}^{(n)} S$ (and $\mathrm{Pat}^{(\ell)} \mathrm{Pat}^{(m)} T = \mathrm{Pat}^{(\ell)} T$).

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Outline

Permutation patterns

The "Galois" connection $\mathrm{Pat}^{(\ell)} - \mathrm{Comp}^{(n)}$

Questions and answers

Further problems

Questions

We shall deal now with the following questions (naturally arising in this context):

- For which $S \subseteq S_\ell$ do we have $\mathrm{Comp}^{(n)} S \leq S_n$?
- Which $T \leq S_n$ are of the form $\mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$? or for some $S \leq S_\ell$?

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The group case for the operator $\mathrm{Comp}^{(n)}$

Proposition

If S is a subgroup of S_ℓ, then $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n.

Sketch of the proof.

Assume that $S \leq S_\ell$. Let $\pi, \tau \in \mathrm{Comp}^{(n)} S$.
Thus $\mathrm{Pat}^{(\ell)} \pi, \mathrm{Pat}^{(\ell)} \tau \subseteq S$.
Crucial observation:

$$\mathrm{Pat}^{(\ell)} \pi^{-1} = (\mathrm{Pat}^{(\ell)} \pi)^{-1} := \{\sigma^{-1} \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi\},$$

$$\mathrm{Pat}^{(\ell)} \pi\tau \subseteq (\mathrm{Pat}^{(\ell)} \pi)(\mathrm{Pat}^{(\ell)} \tau) := \{\sigma\sigma' \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi,\ \sigma' \in \mathrm{Pat}^{(\ell)} \tau\}.$$

Consequently, π^{-1} and $\pi\tau$ also belong to $\mathrm{Comp}^{(n)} S$ (since $S \leq S_\ell$). Thus $\mathrm{Comp}^{(n)} S$ is a group. □

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The group case, continued

The converse of the Proposition does not hold.

There even exist subgroups $H \leq S_n$ which are of the form $\mathrm{Comp}^{(n)}\, S$ ($S \subseteq S_\ell$) but there is no group $G \leq S_\ell$ such that $H = \mathrm{Comp}^{(n)}\, G$.

However: Let $\ell \leq n$ and $\ell \leq 3$. Then we have for $S \subseteq S_\ell$:
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The groups $\mathrm{Comp}^{(n)} S$ are ℓ-closed

Recall the (usual) Galois connection

$$\mathrm{Inv}\, T := \{\varrho \in \mathrm{Rel}_n \mid \forall \pi \in T : \pi \triangleright \varrho\} \text{ for } T \subseteq S_n,$$
$$\mathrm{Aut}\, R := \{\pi \in S_n \mid \forall \varrho \in R : \pi \triangleright \varrho\} \text{ for } R \subseteq \mathrm{Rel}_n\,.$$

Proposition

Assume that $T = \mathrm{Comp}^{(n)} S$ is a subgroup of S_n for some subset $S \subseteq S_\ell$ (not necessarily a subgroup). Then T is ℓ-closed, i.e., $T = \mathrm{Aut}\,\mathrm{Inv}\, T$ is determined by its ℓ-ary invariant relations:

$$T = \mathrm{Aut}\,\mathrm{Inv}^{(\ell)}\, T.$$

In particular, the ℓ-orbits $(h_L)^T$ already characterize the group:

$$T = \mathrm{Aut}\{(h_L)^T \mid L \in \tbinom{[n]}{\ell}\}.$$

where $h_L := (i_1, \ldots, i_\ell)$ for $L = \{i_1, \ldots, i_\ell\} \subseteq [n]$ with $i_1 < \cdots < i_\ell$.

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Characterization theorems

Theorem

Let T be a subgroup of S_n. Then $T = \mathrm{Comp}^{(n)} G$ for some group $G \leq S_\ell$ if and only if $T = \mathrm{Aut}\,\varrho$ for some ℓ-ary irreflexive relation ϱ on $[n]$ that satisfies the condition

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Let $T \leq S_n$, consider the ℓ-orbits $\varrho_L := (h_L)^T$ for all $L \in \binom{[n]}{\ell}$. Then $T = \mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$ if and only if

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Further problems

- What about the (Galois) kernels $\mathrm{Pat}^{(\ell)}\, T$ for $T \subseteq S_n$ or $T \leq S_n$?
- For a given group $G \leq S_\ell$, what can be said about the sequence $G, \mathrm{Comp}^{(\ell+1)}\, G, \ldots, \mathrm{Comp}^{(n)}\, G, \ldots$? Many results are already known (in particular for special groups G, transitive [ATKINSON/BEALS], intransitive, primitive [E. LEHTONEN] ... (ask Erkko))

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Nik Ruškuc, *Classes of permutations avoiding* 231 *or* 321. Lecture given at TU Dresden (*Dresdner Mathematisches Seminar*), Nov. 25, 2015.


Thank you for your ATTENTION!