Algebraic Methods in Quantum Logic

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Preface

This monograph gives an overview of main results obtained within the solution of the project "Algebraic methods in quantum logic", being realized in the cooperation of the Faculty of Science of Masaryk University in Brno and the Faculty of Science of Palacký University in Olomouc. An innovated approach of the studied branch of mathematics consists of several chapters. The first chapter comprises of summarization of the results. The further chapters deal with more detailed and illustrated view of the studied topic.

The primary aim of this monograph is to provide a unified perspective and readable commentary on the studied subject. The work presented here is mainly based on the authors' recent results in the studied field, and the focus of our research has been on the following topics

- Applications of the methods of linear programming in the realm of MV-algebras;
- Applications of the methods of universal algebra in the structural theory of algebras of non-classical logics;
- Application of model theory to algebraic models of logics, theory of partial embeddings and ultrapowers.

The main topic of this monograph is the study of algebraic structures which model non-classical logics. Generally, the presented structures possess an operation $\cdot$, which is usually an alternative to a logical connective "conjunction", and an operation $\to$, which models an implication. The classical concept of conjunction is expressed by the word "and" where the proposition "x and y" is true is usually interpreted as x and y is true together in one place and at the same time. This fact is reflected in the axioms of commutativity

$$x \cdot y = y \cdot x$$

and associativity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

of the respective operation $\cdot$. Firstly, we give some examples of natural conjunctions which are not commutative or associative.

The logical connective "and then" in the sequence of time is clearly non-commutative, the proposition "x and then y" has another meaning than "y and then x". As an example of a non-associative conjunction we can mention a connective "with"[a].

Logical propositions can be naturally ordered by a relation $\leq$ where $x \leq y$ means that the proposition y has bigger truth value then the proposipition x. In other words, $x \leq y$ if and only if $x \to y$ is a tautology. The least element of this order is denoted by 0 and it says to be "absolute false". Dually, the greatest element is denoted by 1 and we call it "absolute truth".

The above mentioned order is fundamental for defining and studying of common properties of operations $\cdot$ and $\to$. More precisely, the operations $\cdot$ and $\to$ satisfy the adjointness property

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z.$$

Non-commutative logics possess two implications[b] $\to$ and $\rightsquigarrow$ and adjointness property has the form

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z \text{ if and only if } y \leq x \rightsquigarrow z.$$

[a]Obviously, the following lunch menu "(chicken with potatoes) with ice-cream" is not the same as "chicken with (potatoes with ice-cream)". The theory of mathematical linguistics works usually with the Lambek calculus which models non-associative conjunctions.

[b]Our example of non-commutative logic is a logic operating with time and conjunction 'and then'. In this case the implications $\to$ and $\rightsquigarrow$ represent an implication to the future and an implication to the past.

This text is divided into five chapters. In Chapter 1 we recall basic definitions and theorems which will be often used in the following parts. We further study some constructions of ultraproducts.

Chapter 2 deals with MV-algebras. We present a direct proof of Di Nola's representation theorem for MV-algebras and extend his results to the restriction of the standard MV-algebra to rational numbers. The results are based on a direct proof of the theorem stating that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into $\mathbb{Q} \cap [0, 1]$. In the next section, we introduce tense MV-algebras which are MV-algebras expanded by new unary operators G and H expressing universal time quantifiers. Analogously to classical works on Boolean algebras, we prove that any tense semisimple MV-algebra is induced by a time frame.

In Chapter 3 we treat a few recent problems arising in the area of lattice effect algebras that generalize both MV-algebras and orthomodular lattices. First, we axiomatize finitely generated varieties of distributive lattice effect algebras. Second, for any positive integer n, the free n-generator algebras in these varieties are described. Third, we present a construction of tense operators G and H in effect algebras to represent the dynamics of formally desribed physical systems. We firstly derive a time frame which enables these constructions and then apply a suitable set of states of the lattice effect algebra $\mathbf{E}$ in question. To approximate physical reality in quantum mechanics, we use only the so-called Jauch-Piron states on $\mathbf{E}$.

Non-associative logics are studied in Chapter 4. In the first section we focus on commutative basic algebras which are a non-associative generalization of MV-algebras. It was proved by P. Wojciechowski [145] that for any infinite cardinal there exists a linearly ordered MV-algebra of this cardinality. This rises a natural question if this is true also for basic algebras which are not MV-algebras. Using P. Wojciechowski's construction and the modified construction of M. Botur [17], we can set up certain defectors which enable us to construct a subdirectly irreducible commutative basic algebra of any infinite cardinality. In the second section we introduce states on commutative basic algebras and present their basic properties. States are defined in the same way as Mundici's states on MV-algebras as normalized finitely additive $[0, 1]$-valued functions, and some results analogous to the results that are known for MV-algebras are proved. The last goal of the chapter is to present a non-associative generalization of Hájek's BL-logic which has the class $naBL$ of non-associative BL-algebras as its algebraic semantics. Moreover, it is shown that $naBL$ forms a variety generated just by non-associative t-norms. Consequently, the non-associative BL-logic is the logic of non-associative t-norms and their residua.

In the last chapter we study the general concept of state-morphisms on an algebra. We present a complete characterization of subdirectly irreducible state BL-algebras as well as of subdirectly irreducible state-morphism BL-algebras. In addition, we present a general theory of state-morphism algebras, that is, algebras of general type with a state-morphism which is an idempotent endomorphism. We define a diagonal state-morphism algebra and show that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one. We describe generators of varieties of state-morphism algebras, in particular, state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MV-algebras.

The reader is assumed to be familiar with the basics of universal algebra [27], the theory of ordered sets and lattices in quantum structures [103] and [73], the algebraization of logics [85] and, in particular, the model theory [44]. This monograph is intented mostly for researchers and advanced graduate students working in the theory of algebraic models of non-classical logics and residuated algebras. Hopefully, the book will be useful for all readers interested in the presented topic.

The book can be used for self-study, as a research seminar text (accompanied by selected readings from the referred literature), or as a supplement of a comprehensive graduate-course on Ordered algebraic structures given at Masaryk University in Brno.

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Contents

Chapter 1

Introduction

1.1 Basic definitions

By *a partial order* on a set A we mean a binary relation $\leq$ on A which is reflexive ($x \leq x$ for all $x \in A$), antisymmetric ($x \leq y$ and $y \leq x$ imply $x = y$, for all $x, y \in A$) and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in A$). We say that the couple $\mathbf{A} = (A; \leq)$ is an *ordered set* (shortly *poset*).

The elements a, b are said to be *comparable* with each other if either $a \leq b$ or $b \geq a$ (or both). Otherwise we write $a \parallel b$ and call a and b *incomparable* with each other. The poset $(A; \leq)$ is called a *chain* if any two of its elements are comparable with each other. If any two different elements of A are not comparable with each other then $(A; \leq)$ is called an *antichain.*

If $a \leq b$ and $a \neq b$, we will write $a < b$. In case $a \leq b$ we will also use the notation $b \geq a$. If $b \geq a$ and $b \neq a$ we express this by the notation $b > a$.

The element a is called *maximal* if there is no $x \in A$ with $a < x$. The element a is called the *greatest element* of $(A; \leq)$ if $y \leq a$ for all $y \in A$. The notions *minimal* respectively *least* or *smallest element* of $(A; \leq)$ are defined dually.

From now on, if $(A;\leq)$ has a least or greatest element, respectively, this element will be denoted by 0 respectively 1. If $(A;\leq)$ is a poset with 0 then a is called an *atom* of $(A;\leq)$ if $a \neq 0$ and there does not exist a $c \in A$ with $O < c < a$. Dually, the notion of a coatom is defined.

If $M \subseteq A$ then $U(M)$ and $L(M)$ denote the set of all upper and lower bounds of M, respectively, i.e.

$$U(M) := \{x \in A \mid m \leq x \text{ for all } m \in M\},$$
$$L(M) := \{x \in A \mid x \leq m \text{ for all } m \in M\}.$$

The sets $U(M)$ and $L(M)$ are called the *upper* and *lower cone* of M, respectively. In case $M = \{a_1, \ldots, a_n\}$ we simply write $U(a_1, \ldots, a_n)$ and $L(a_1, \ldots, a_n)$ instead of $U(M)$ and $L(M)$, respectively. For $a \in A$ the cone $U(a)$ and $L(a)$ is called the *principal order filter* and the *principal order ideal* generated by a, respectively. An *order ideal* of $(A;\leq)$ is a non-void subset I of A such that $a \in A$, $b \in I$ and $a \leq b$ together imply $a \in I$.

The poset $(A;\leq)$ is called *up-directed* or *down-directed* if for all $a, b \in A$ we have $U(a,b) \neq \emptyset$ or $L(a,b) \neq \emptyset$, respectively. $(A;\leq)$ is called *directed* if it is both up- and down-directed. Obviously, a poset with 1 is up-directed and a poset with 0 down-directed. A poset is called *bounded* if it has 0 and 1. Hence every bounded poset is directed.

If for a subset M of A the subset $U(M)$ has a smallest element then this is called the *supremum* and it is denoted by $\sup(M)$ or $\bigvee M$. Dually the notion of the *infimum* $\inf(M)$ or $\bigwedge M$ of M is defined. In case of $M = \{a_1, \ldots, a_n\}$ we simply write $\sup(a_1, \ldots, a_n)$ or $a_1 \vee \cdots \vee a_n$ and $\inf(a_1, \ldots, a_n)$ or $a_1 \wedge \cdots \wedge a_n$, respectively.

A mapping $f : A^n \longrightarrow A$ is called an *n-nary operation*, the number n is the *arity* of the operation. An *algebra* is a structure $\mathbf{A} = (A; F)$, where A is an arbitrary non-empty set and F is a system of operations. *A type* of algebra is a mapping from F to $\mathbb{N}$ (natural numbers including zero) which maps any $f \in F$ to its arity. If the set F is finite, then the algebra $\mathbf{A} = (A; F)$ can be written also as $\mathbf{A} = (A; f_1, \ldots, f_k)$ where $F = \{f_1, \ldots, f_k\}$. The *type of the algebra* $\mathbf{A}$ is the vector $\langle n_1 \ldots, n_k\rangle \in \mathbb{N}^k$ where f_i is just n_i-nary operation.

In what follows we will often work with partially ordered sets which possess finite suprema and finite infima.

1.1.1 Semilattices, lattices and Boolean algebras.

Definition 1.1. An algebra $\mathbf{S} = (S; \cdot)$ of type $\langle 2 \rangle$ is called a *groupoid.*

An algebra $\mathbf{S} = (S; \cdot)$ of type $\langle 2 \rangle$ is called *a semigroup* if it satisfies for any $x, y, z \in S$

(ASS) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

A *monoid* is an algebra $\mathbf{S} = (S; \cdot, e)$ of type $\langle 2, 1 \rangle$, where $(S; \cdot)$ is a semigroup and it satisfies for any $x \in S$

(IDEN) $x \cdot e = x$ and $e \cdot x = x$.

A semigroup $\mathbf{S} = (S; \cdot)$ is called *commutative* if it satisfies for any $x, y \in S$

(COMM) $x \cdot y = y \cdot x$.

A commutative semigroup $\mathbf{S} = (S; \vee)$ is called *a (join) semilattice* if it satisfies for any $x \in S$

(IDEM) $x \vee x = x$.

A lattice is an algebra $\mathbf{L} = (L; \vee, \wedge)$ of type $\langle 2, 2 \rangle$, where $(L; \vee)$ and $(L; \wedge)$ are semilattices and

(LAT) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$

holds for any $x, y \in L$.

If $\mathbf{S} = (S; \vee)$ is a semilattice then there is an *order* $\leq$ *induced by semilattice* $\mathbf{S}$ on the set S defined by $x \leq y$ if and only if $x \vee y = y$. Moreover, $x \vee y$ is the supremum of x and y, more precisely, $x, y \leq x \vee y$ and $x, y \leq z$ implies $x \vee y \leq z$.

If $\mathbf{L} = (L; \vee, \wedge)$ is a lattice, then the order $\leq$ induces by the semilattice $(L; \vee)$ is same as the order induced by the semilattice $(L; \wedge)$, thus

$$x \leq y \text{ if and only if } x \vee y = y \text{ if and only if } x \wedge y = x.$$

The operations $\vee$ and $\wedge$ represent suprema and infima and we call them *join* and *meet*.

Both semilattices and lattices can be considered as structures with a "double face", i.e. as posets on the one hand or as algebras with one or two binary operations on the other hand. We usually prefer to work with their representation as algebras since this enables us to apply the machinery developed in universal algebra.

A *bounded lattice* is an algebra $\mathbf{L} = (L; \vee, \wedge, 0, 1)$ of type $\langle 2, 2, 0, 0\rangle$ where $(L; \vee, \wedge)$ is a lattice with least element 0 and greatest element 1 with respect to the induced order.

A lattice $\mathbf{L}$ is *distributive* if it satisfies

$$\text{(LD)} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

or equivalently (see [27])

$$(\text{LD}^{op}) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice $\mathbf{L}$ is called *modular* if it satisfies the quasi-identity

$$x \leq z \text{ implies } (x \vee y) \wedge z = x \vee (y \wedge z)$$

which is equivalent to the *modular identity*

$$\text{(MOD)} \quad (x \vee y) \wedge (x \vee z) = x \vee (y \wedge (x \vee z)).$$

It can be shown that a lattice is modular if and only if it satisfies one of the two distributive laws for all triples of elements where two of the three elements are comparable with each other.

A mapping f from A to A is called *isotone* or *monotone* if $x, y \in A$ and $x \leq y$ together imply $f(x) \leq f(y)$, *antitone* if $x, y \in A$ and $x \leq y$ together imply $f(y) \leq f(x)$ and an *involution* if $f(f(x)) = x$ for each $x \in A$.

Let $\mathbf{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice. Then we say that $y \in L$ is a complement of an element $x \in L$ if it satisfies $x \vee y = 1$ and $x \wedge y = 0$. We remark that in a distributive lattice there is at most one complement to any element.

We say that $\mathbf{L} =$ is a *lattice with an antitone involution* if there is a mapping $x \mapsto x'$ of L into itself such that $x'' = x$ and $x \leq y \Rightarrow y' \leq x'$.

If $\mathbf{L} =$ is a lattice with an antitone involution then it satisfies the so-called De Morgan Laws:

$$(x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'$$

If $\mathbf{L} =$ is a bounded lattice with complementation which is simultaneously an antitone involution then the system $(L; \vee, \wedge, ', 0, 1)$ is called an *ortholattice.*

An ortholattice $(L; \vee, \wedge, ', 0, 1)$ is called an *orthomodular* (briefly OML) if it satisfies the quasiidentity

(OMQ) $x \leq y \Rightarrow x \vee (x' \wedge y) = y$

which is equivalent to the identity

(OMI) $x \vee (x' \wedge (x \vee y)) = x \vee y.$

Let us note that x' need not be the unique complement of x.

Definition 1.2. An algebra $\mathbf{L} = (L; \vee, \wedge, \neg, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$ is *a complemented lattice* if $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and for every $x \in A$ $\neg x$ is a complement of x. A distributive complemented lattice is called a *Boolean algebra.*

We remark that if X is an arbitrary set and $\wp(X)$ is the power set of X then $(\wp(X); \cup, \cap, {}^c, \emptyset, X)$, where $Y^c := X \setminus Y$ for any $Y \in \wp(X)$, is a Boolean algebra. The class of Boolean algebras is just the algebraic counterpart of classical propositional logic.

It is easy to see that every Boolean algebra is an orthomodular lattice and hence an ortholattice. Conversely, an orthomodular lattice is a Boolean algebra if and only if it is distributive. For the proofs and details, the reader is referred to books [6] or [103], [8], [93].

Definition 1.3. A structure $\mathbf{S} = (S; \cdot, \to, \rightsquigarrow, \leq)$ is called a *residuated groupoid* if:

(RES1) $(S; \cdot)$ is a groupoid and $(S; \leq)$ is an poset.

(RES2) $(S; \cdot, \to, \rightsquigarrow)$ is an algebra of type $\langle 2, 2, 2\rangle$.

(RES3) The operations $\cdot$, $\to$ and $\rightsquigarrow$ satisfy the adjointness property

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z \text{ if and only if } y \leq x \rightsquigarrow z.$$

If $(S; \cdot)$ is a commutative groupoid then $\to = \rightsquigarrow$ and we say that $\mathbf{S} = (S; \cdot, \to, \leq)$ is a *commutative residuated groupoid.*

If $(S; \cdot)$ is a semigroup we say that $\mathbf{S}$ is a *residuated poset.*

An algebra $\mathbf{S} = (S; \vee, \wedge, \cdot, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ is said to be a *non-associative residuated lattice* if $(S; \vee, \wedge, 0, 1)$ is a bounded lattice and $(S; \cdot, \to, \leq)$ is a commutative residuated groupoid where $\leq$ is the induced order with respect to lattice operations $\vee$ and $\wedge$ and 1 is a neutral element of the groupoid $(S; \cdot)$.

1.1.2 Basic remarks on universal algebra

Let us have an algebra $\mathbf{A} = (A; \mathrm{F})$. Then an algebra $\mathbf{B} = (B; \mathrm{F})$ of the same type such that $B \subseteq A$ and the algebraic operations of $\mathbf{A}$ are restricted to B is called a *subalgebra* of $\mathbf{A}$ and we write $\mathbf{B} \subseteq \mathbf{A}$. If $\mathcal{K}$ is a class of algebras then $\mathrm{S}(\mathcal{K})$ denotes the class of all subalgebras of algebras from the class $\mathcal{K}$.

Let us have a system of algebras $\{\mathbf{A}_i = (A_i; \mathrm{F}) \mid i \in I\}$. Then the *direct product of algebras* $\mathbf{A}_i$ is the algebra $\prod_{i \in i} \mathbf{A}_i = (\prod_{i \in I} A_i; \mathrm{F})$ where

$$f(x_1 \ldots, x_n)(i) := f(x_1(i), \ldots, x_n(i))$$

for any n-nary operation $f \in \mathrm{F}$ and any $x_1, \ldots, x_n \in \prod_{i \in I} A_i$. The class of all direct products of algebras from a class $\mathcal{K}$ is denoted by $\mathrm{P}(\mathcal{K})$.

Let us have algebras $\mathbf{A} = (A; \mathrm{F})$ and $\mathbf{B} = (B; \mathrm{F})$. Then a mapping $\phi : A \longrightarrow B$ is called *a homomorphism* if for any n-nary operation $f \in \mathrm{F}$ and any $x_1, \ldots, x_n \in A$ it satisfies

$$\phi(f(x_1, \ldots, x_n)) = f(\phi(x_1), \ldots, \phi(x_n)).$$

If the mapping ϕ is surjective, then we say that $\mathbf{B}$ is *a homomorphic image* of $\mathbf{A}$. The class of all homomorphic images of algebras from $\mathcal{K}$ is denoted by $\mathrm{H}(\mathcal{K})$.

An equivalence θ on a set A is called *a congruence of an algebra* $\mathbf{A} = (A; \mathrm{F})$ if for any n-nary operation $f \in \mathrm{F}$ and $x_1, y_1, \ldots, x_n, y_n \in A$

$$\text{whenever } x_i \theta y_i \text{ for all } i = 1, \ldots, n, \text{ then } f(x_1, \ldots, x_n) \theta f(y_1, \ldots, y_n).$$

Let us have a congruence θ on an algebra $\mathbf{A} = (A; \mathrm{F})$. Then we define *the factor algebra* $\mathbf{A}/\theta = (A/\theta; \mathrm{F})$, where

$$x/\theta := \{y \in A \mid x\theta y\},$$

$$A/\theta := \{x/\theta \mid x \in A\}$$

and

$$f(x_1/\theta, \ldots, x_n/\theta) := f(x_1, \ldots, x_n)/\theta$$

for any n-nary operation $f \in \mathrm{F}$ and any $x_1, \ldots, x_n \in A$. The set of all congruences of the algebra $\mathbf{A}$ is denoted by $\mathrm{Con}\,\mathbf{A}$. $\mathrm{Con}\,\mathbf{A}$ is a complete lattice with respect to set-inclusion.

It can be easily checked that the mapping $x \mapsto x/\theta$ is a surjective homomorphism, thus $\mathbf{A}/\theta \in \mathrm{H}(\mathbf{A})$. Conversely, if $f : \mathbf{A} \longrightarrow \mathbf{B}$ is a surjective homomorphism, then

$$\theta_f := \{(x, y) \in A^2 \mid f(x) = f(y)\} \in \mathrm{Con}\,\mathbf{A}$$

and $\mathbf{B} \cong \mathbf{A}/\theta_f$ holds.

A class $\mathcal{K}$ of algebras of the same type is *a variety* if $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$, $\mathrm{P}(\mathcal{K}) \subseteq \mathcal{K}$ and $\mathrm{H}(\mathcal{K}) \subseteq \mathcal{K}$ hold. A class $\mathcal{K}$ is variety if and only if it satisfies $\mathcal{K} = \mathrm{HSP}(\mathcal{K})$ (see [27]). Moreover, $\mathrm{HSP}(\mathcal{K})$ is the smallest variety which contains $\mathcal{K}$. If $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$ then we say that *the variety $\mathcal{V}$ is generated by the class $\mathcal{K}$.*

A class $\mathcal{K}$ of algebras of the same type is *a quasivariety* if is the class of all models of a set of quasiidentities, that is, implications of the form $s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \rightarrow s \approx t$, where $s, s_1, \ldots, s_n, t, t_1, \ldots, t_n$ are terms built up from variables using the operation symbols of the specified signature.

1.1.3 Commutative and associative logics

Petr Hájek (see [97]) introduced the class of basic logic algebras (briefly BL-algebras).

Definition 1.4. *An algebra* $\mathbf{B} = (B; \vee, \wedge, \cdot, \to, 0, 1)$ *of type* $\langle 2, 2, 2, 2, 0, 0\rangle$ *is called a* BL-algebra *if:*

- *(BL1)* $(B; \cdot, 1)$ *is a commutative monoid.*
- *(BL2)* $(B; \vee, \wedge, 0, 1)$ *is a bounded lattice with an order* $\leq$*.*
- *(BL3)* $(B; \cdot, \to, \leq)$ *is a commutative residuated poset.*
- *(BL4) The axiom of divisibility* $(x \to y) \cdot x = x \wedge y$ *holds.*
- *(BL5) The axiom of prelinearity* $(x \to y) \vee (y \to x) = 1$ *holds.*

The class of BL-algebras is a generalization of the class of Boolean algebras.

Example 1.1. *Let us have a Boolean algebra* $\mathbf{B} = (B; \vee, \wedge, \neg, 0, 1)$*. If we define* $x \cdot y := x \wedge y$ *and* $x \to y := \neg x \vee y$*, then the algebra* $(B; \vee, \wedge, \cdot, \to, 0, 1)$ *is a BL-algebra.*

Very important examples of BL-algebras are BL-algebras on the interval $[0, 1]$ of real numbers induced by t-norms.

Definition 1.5. *A binary operation* $*$ *on the interval* $[0, 1]$ *is a continuous t-norm if:*

- *(t1)* $([0, 1]; *, 1)$ *is a commutative monoid.*
- *(t2) The operation* $*$ *is continuous as a function, in the usual interval topology on* $[0, 1]^2$*.*
- *(t3) The operation* $*$ *is monotone, thus if* $x \leq y$ *then* $x * z \leq y * z$ *for any* $x, y, z \in [0, 1]$*.*

There are three basic examples of t-norms:

$$x * y := \min\{x, y\}, \tag{1.1}$$

$$x * y := \max\{0, x + y - 1\}, \tag{1.2}$$

$$x * y := xy. \tag{1.3}$$

The t-norm (1.1) is called Gödel's t-norm, (1.2) is Łukasiewicz's t-norm and (1.3) is the product t-norm. It is proved in [97] how we can obtain any t-norm by composing these three ones. The following theorem (see [47], [97]) shows the relation between t-norms and BL-algebras.

Theorem 1.1. *If* $*$ *is a t-norm then the algebra* $([0,1]; \max, \min, *, \rightarrow_*, 0, 1)$, *where*

$$x \rightarrow_* y := \max\{z \mid x * z \leq y\},$$

is a BL-algebra.

The class of BL-algebras form a variety. This variety is generated by the set of BL-algebras induced by t-norms.

Congruences of a BL-algebra can be described by the kernels (class which contains the element 1). Consequently, the lattice of congruences of an arbitrary BL-algebra is isomorphic to the lattice of filters.

Definition 1.6. *Let* $\mathbf{B} = (B; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *be a BL-algebra. Then a set* $F \subseteq A$ *is* a filter *if*

(F1) $1 \in F$,

(F2) if $x \in F$ *and* $x \leq y$ *then* $y \in F$,

(F3) if $x, y \in F$ *then* $x \cdot y \in F$.

The correspondence between filters and congruences of BL-algebras is described in the following theorem.

Theorem 1.2. *Let us have a BL-algebra* $\mathbf{B} = (B; \vee, \wedge, \cdot, \rightarrow, 0, 1)$.

i) If $\theta \in \operatorname{Con} \mathbf{B}$ *then the set* $1/\theta$ *is a filter.*

ii) If F *is a filter of* $\mathbf{B}$ *then*

$$\theta_F = \{(x, y) \mid x \rightarrow y, y \rightarrow y \in F\}$$

is the only congruence of $\mathbf{B}$ *such that* $F = 1/\theta_F$.

For simplicity, we replace x/θ_F and $\mathbf{B}/\theta_F$ by x/F and $\mathbf{B}/F$, respectively.

This correspondence allows us to study properties of homomorphisms and congruences by means of filters.

A filter F on a BL-algebra $\mathbf{B} = (B; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ is called *prime* if it satisfies one of the following pairwise equivalent conditions:

i) $x \vee y = 1$ implies $x \in F$ or $y \in F$, for all $x, y \in F$.

ii) $x \rightarrow y \in F$ of $y \rightarrow x \in F$, for all $x, y \in F$.

iii) $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in F$.

iv) $\mathbf{B}/F$ is linearly ordered.

For any $x \in B$ and any filter $F \subseteq B$ such that $x \notin F$, there is a prime filter P such that $F \subseteq P$ and $x \notin P$. Consequently, any BL-algebra $\mathbf{B}$ is subdirect product $\prod\{\mathbf{B}/P \mid P \text{ is a prime filter}\}$ (see [97]).

Maximal proper filters of BL-algebra are called *ultrafilters* and they are prime.

Let us have a Boolean algebra $\mathbf{B} = (B; \vee, \wedge, \neg, 0, 1)$. A *filter* $F \subseteq B$ of $\mathbf{B}$ is a non-empty subset closed with respect to meets and upper ends. A maximal filter (which does not contain 0) is called an *ultrafilter*.

If $V \subseteq B$ is a subset such that for any finite $X \subseteq V$ we have $\bigwedge X \neq 0$ then there is an ultrafilter U such that $V \subseteq U$ (so called *finite intersection property*). Moreover, if $x \in B$ and U is an ultrafilter then either $x \in U$ or $\neg x \in U$.

If I is a set and $\wp(I)$ its power set then the algebra $(\wp(I); \cup, \cap, {}^c, \emptyset, I)$ is a Boolean algebra. Thus an (ultra)filter of $\wp(I)$ can be interpreted as an (ultra)filter of this Boolean algebra. If F is an filter of $\wp(I)$ we say that F is a filter over I.

In case of Boolean algebras, ultrafiters are just prime filters. Moreover, the only linearly ordered Boolean algebra is two element one. Thus any Boolean algebra is a subdirect product of two element Boolean algebra.

1.2 General finite embeddability theorem

Let $\{\mathbf{A}_i \mid \; i \in I\}$ be a system of algebras of the same type. We denote for any $x, y \in \prod_{i\in I} A_i$ the sets

$$\llbracket x = y \rrbracket = \{j \in I \mid x(j) = y(j)\},$$
$$\llbracket x \neq y \rrbracket = \{j \in I \mid x(j) \neq y(j)\}.$$

If F is a filter of $\wp(I)$ then the relation θ_F defined by

$$\theta_F = \{(x, y) \in (\prod_{i\in I} A_i)^2 \mid \llbracket x = y \rrbracket \in F\}$$

is a congruence on $\prod_{i\in I} \mathbf{A}_i$. Moreover, for an ultrafilter U of $\wp(I)$, the algebra $(\prod_{i\in I} \mathbf{A}_i)/U := (\prod_{i\in I} \mathbf{A}_i)/\theta_U$ is said to be an *ultraproduct* of algebras $\{\mathbf{A}_i \mid \; i \in I\}$. The class of all ultraproducts of algebras from $\mathcal{K}$ is denoted by $\mathrm{P_U}(\mathcal{K})$.

It can be easily checked that just one of the sets $\llbracket x = y \rrbracket$ and $\llbracket x \neq y \rrbracket$ belongs to any ultrafilter U.

Recall that a class $\mathcal{K}$ is a quasivariety iff it contains a trivial algebra and is closed under isomorphisms, subalgebras, direct products, and ultraproducts (see [27]).

Definition 1.7. Let $\mathbf{A} = (A, \mathrm{F})$ be an algebra, $X \subseteq A$ and $\mathrm{F} \subseteq \mathrm{E}$. A *partial subalgebra* is a pair $\mathbf{X} = (X, \mathrm{E})$, where for any $f \in E_n$ and all $x_1, \dots, x_n \in X$, $f^{\mathbf{X}}(x_1, \dots, x_n)$ is defined if and only if $f^{\mathbf{A}}(x_1, \dots, x_n) \in X$. We put then

$$f^{\mathbf{X}}(x_1, \dots, x_n) := f^{\mathbf{A}}(x_1, \dots, x_n).$$

If $\mathrm{F} = \mathrm{E}$ we speak about a *partial algebra* $\mathbf{A}|_X$. Denote by $\mathrm{S}_{\mathrm{P}(\mathcal{K})}$ the class of all partial subalgebras of algebras from the class $\mathcal{K}$.

We write $\mathcal{K}_{\mathrm{F}}$ to indicate that $\mathcal{K}$ is a class of algebras of type F.

Definition 1.8. A partial algebra $\mathbf{A} = (A; \mathrm{F})$ satisfies *the general finite embeddability* (*finite embeddability property*) *property for the class* $\mathcal{K}$ of algebras of the same type if for any finite subset $X \subseteq A$, there exist a (finite) algebra $\mathbf{B} \in \mathcal{K}_{\mathrm{E}}$ and an embedding $\rho : \mathbf{A}|_X \hookrightarrow \mathbf{B}$, i.e., an injective mapping $\rho : X \to B$ satisfying the property $\rho(f^{\mathbf{A}|_X}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \ldots, \rho(x_n))$ if $x_1, \ldots, x_n \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_X}(x_1, \ldots, x_n)$ is defined.

Finite embeddability property is usually denoted (FEP). Note also that

(i) a quasivariety $\mathcal{K}$ has FEP (i.e., any element of $\mathcal{K}$ has FEP) if and only if $\mathcal{K} = \mathrm{ISPP_U}(\mathcal{K}_{Fin})$ (see [9, Theorem 1.1] or [10]),

(ii) FEP yields the general finite embeddability property.

Theorem 1.3. *Let* $\mathbf{A} = (A; \mathrm{F})$ *be a partial algebra and let* $\mathcal{K}_{\mathrm{E}}$ *be a class such that* $\mathrm{F} \subseteq \mathrm{E}$*. If* $\mathbf{A}$ *satisfies the general finite embeddability property for* $\mathcal{K}_{\mathrm{E}}$ *then* $\mathbf{A} \in \mathrm{IS_P P_U}(\mathcal{K}_{\mathrm{F}})$*.*

Proof. Let $\mathbf{A}$ satisfy the general finite embeddability property for $\mathcal{K}_{\mathrm{E}}$. Denote $I = \{X \subseteq A \mid X \text{ is finite}\}$. Then for any $X \in I$ there are $\mathbf{A}_X \in \mathcal{K}_{\mathrm{E}}$ and an embedding $\rho_X : \mathbf{A}|_X \to \mathbf{A}_X$. By axiom of choice we choose a fixed $a_X \in A_X$ for any $X \in I$. Now we define a mapping $\varphi : \mathbf{A} \to \prod_{X \in I} \mathbf{A}_X$ by

$$\varphi(a)(X) = \begin{cases} \rho_X(a) & \text{if } a \in X \\ a_X & \text{otherwise.} \end{cases}$$

Denote further $U(X) = \{Y \in I \mid X \subseteq Y\}$ and $V = \{U(X) \mid X \in I\}$. Then for any $U(X), U(Y) \in V$ the equality $U(X) \cap U(Y) = U(X \cup Y)$ holds and thus $U(X) \cap U(Y) \neq \emptyset$, $U(X \cup Y) \in V$. Consequently there is an ultrafilter U of $\wp(I)$ such that $V \subseteq U$. Hence, we can define a mapping

$$\rho : \mathbf{A} \to (\prod_{X \in I} \mathbf{A}_X)/U$$

such that $\rho(a) = \varphi(a)/U$.

(i) ρ *is injective.* Let $x, y \in A$ be such that $x \neq y$. Then for any $X \in U(\{x,y\})$ we have $\varphi(x)(X) = \rho_X(x) \neq \rho_X(y) = \varphi(y)(X)$. Hence $U(\{x,y\}) \subseteq [\![\varphi(x) \neq \varphi(y)]\!] \in U$ and finally $\rho(x) = \varphi(x)/U \neq \varphi(y)/U = \rho(y)$.

(ii) ρ *is a homomorphism.* Take $f \in \mathrm{F}_n$ and $x_1, \ldots, x_n \in A$ such that $f^{\mathbf{A}}(x_1, \ldots, x_n)$ is defined. For any $X \in U(\{x_1, \ldots, x_n, f^{\mathbf{A}}(x_1, \ldots, x_n)\})$, we have

$$\begin{array}{rcl} \varphi(f^{\mathbf{A}}(x_1, \ldots, x_n))(X) & = & \rho_X(f^{\mathbf{A}}(x_1, \ldots, x_n)) \\ & = & f^{\mathbf{A}_X}(\rho_X(x_1), \ldots, \rho_X(x_n)) \\ & = & f^{\mathbf{A}_X}(\varphi(x_1)(X), \ldots, \varphi(x_n)(X)) \\ & = & f^{\prod_{Y \in I} \mathbf{A}_Y}(\varphi(x_1), \ldots, \varphi(x_n))(X). \end{array}$$

Hence,

$$U(\{x_1, \ldots, x_n, f^{\mathbf{A}}(x_1, \ldots, x_n)\}) \subseteq$$
$$[\![\varphi(f^{\mathbf{A}}(x_1, \ldots, x_n)) = f^{\prod_{Y \in I} \mathbf{A}_Y}(\varphi(x_1), \ldots, \varphi(x_n))]\!] \in U$$

holds. Now we compute

$$\begin{array}{rcl} \rho(f^{\mathbf{A}}(x_1, \ldots, x_n)) & = & \varphi(f^{\mathbf{A}}(x_1, \ldots, x_n))/U \\ & = & f^{\prod_{Y \in I} \mathbf{A}_Y}(\varphi(x_1), \ldots, \varphi(x_n))/U \\ & = & f^{(\prod_{Y \in I} \mathbf{A}_Y)/U}(\varphi(x_1)/U, \ldots, \varphi(x_n)/U) \\ & = & f^{(\prod_{Y \in I} \mathbf{A}_Y)/U}(\rho(x_1), \ldots, \rho(x_n)). \end{array}$$

This shows that ρ is an embedding and $\mathbf{A} \in \mathrm{IS_P P_U}(\mathcal{K}_\mathrm{F})$. $\square$

Theorem 1.4. *Let $\mathbf{A} = (A; \mathrm{F})$ be a partial algebra such that F is finite and let $\mathcal{K}_\mathrm{E}$ be a class such that $\mathrm{F} \subseteq \mathrm{E}$. If $\mathbf{A} \in \mathrm{IS_P P_U}(\mathcal{K}_\mathrm{F})$ then $\mathbf{A}$ satisfies the general finite embeddability property for $\mathcal{K}_\mathrm{E}$.*

Proof. If $\mathbf{A} \in \mathrm{IS_P P_U}(\mathcal{K}_\mathrm{F})$ holds then there are $\mathbf{A}_i \in \mathcal{K}_\mathrm{F}$ for any $i \in I$, an ultrafilter U of $\wp(I)$ and a partial subalgebra $\mathbf{A}'$ of the algebra $(\prod_{i\in I} \mathbf{A}_i)/U$ such that $\mathbf{A} \cong \mathbf{A}'$ holds. Take a finite set $X \subseteq A' \subseteq (\prod_{i\in I}(A_i)/U$. Further, for any $x \in X$, choose a fixed element $x' \in \prod_{i\in I} A_i$ such that $x = x'/U$ (in other words, we choose a fixed x' such that $x' \in x$ holds). Now define the sets

$$V = \{[\![f^{\prod_{i\in I}\mathbf{A}_i}(x_1', \ldots, x_n') = x']\!] \mid x_1, \ldots, x_n, x \in X,\ f^{\mathbf{A}'}(x_1, \ldots, x_n) = x\},$$

$$W = \{[\![x' \neq y']\!] \mid\ x, y \in X \text{ such that } x \neq y\}.$$

If $x_1, \ldots, x_n, x \in X$ are such that $f^{\mathbf{A}'}(x_1, \ldots, x_n) = x$ for $f \in F_n$ then we have

$$\begin{aligned} x'/U &= x \\ &= f^{\mathbf{A}'}(x_1, \ldots, x_n) \\ &= f^{(\prod_{i\in I}\mathbf{A}_i)/U}(x_1'/U, \ldots, x_n'/U) \\ &= f^{\prod_{i\in I}\mathbf{A}_i}(x_1', \ldots, x_n')/U. \end{aligned}$$

This shows $[\![f^{\prod_{i\in I}\mathbf{A}_i}(x_1', \ldots, x_n') = x']\!] \in U$ and thus $V \subseteq U$. Moreover, one can easily check that the finiteness of both X and F yield the finiteness of V.

Analogously, if any $x, y \in X$ satisfy $x \neq y$ then $x'/U \neq y'/U$ holds, which yields $[\![x' \neq y']\!] \in U$. Hence, $W \subseteq U$ and finiteness of X entail finiteness of W.

Since V and W are finite and both are the subsets of the ultrafilter U, also $\bigcap V \cap \bigcap W \in U$ and thus $\bigcap V \cap \bigcap W \neq \emptyset$. Then there exists $j \in \bigcap V \cap \bigcap W$. Now define a mapping $\rho : X \to A_j$ such that $\rho(x) = x'(j)$ for any $x \in X$.

(i) *ρ is injective.* Let $x, y \in X$ be such that $x \neq y$. The property $j \in \bigcap V \cap \bigcap W \subseteq \bigcap W \subseteq [\![x' \neq y']\!]$ gives $\rho(x) = x'(j) \neq y'(j) = \rho(y)$.

(ii) ρ *is a homomorphism.* Choose $f \in F_n$ and any $x_1, \ldots, x_n, x \in X$ such that $x = f^{\mathbf{A}'}(x_1, \ldots, x_n)$ with

$$j \in \bigcap V \cap \bigcap W \subseteq \bigcap V \subseteq [\![f^{\prod_{i \in I}(\mathbf{A}_i)}(x'_1, \ldots, x'_n) = x']\!].$$

This yields $f^{\prod_{i \in I}(\mathbf{A}_i)}(x'_1, \ldots, x'_n)(j) = x'(j)$ and

$$\begin{aligned} \rho(f^{\mathbf{A}'}(x_1, \ldots, x_n)) &= \rho(x) \\ &= x'(j) \\ &= f^{\prod_{i \in I} \mathbf{A}_i}(x'_1, \ldots, x'_n)(j) \\ &= f^{\mathbf{A}_j}(x'_1(j), \ldots, x'_n(j)) \\ &= f^{\mathbf{A}_j}(\rho(x_1), \ldots, \rho(x_n)). \end{aligned}$$

Altogether, we have proved that the algebra $\mathbf{A}'$ satisfies the general finite embeddability property and $\mathbf{A} \cong \mathbf{A}'$ that proves the theorem. $\square$

Corollary 1.1. *Let* $\mathbf{A} = (A; \mathrm{F})$ *be a partial algebra and let* $\mathcal{K}_{\mathrm{F}}$ *be a class of algebras such that* F *is a finite set. Then* $\mathbf{A}$ *satisfies the general finite embeddability property in* $\mathcal{K}_{\mathrm{F}}$ *if and only if* $\mathbf{A} \in \mathrm{IS}_{\mathrm{P}}\mathrm{P}_{\mathrm{U}}(\mathcal{K}_{\mathrm{F}})$.

Corollary 1.2. *Let* $\mathbf{A} = (A; \mathrm{F})$ *be an algebra and let* $\mathcal{K}_{\mathrm{F}}$ *be a class of algebras such that* F *is a finite set. Then* $\mathbf{A}$ *satisfies the general finite embeddability property in* $\mathcal{K}_{\mathrm{F}}$ *if and only if* $\mathbf{A} \in \mathrm{ISP}_{\mathrm{U}}(\mathcal{K}_{\mathrm{F}})$.

Proof. Clearly, if $\mathbf{A}$ is an algebra then $A \in \mathrm{S}(\mathcal{K}_{\mathrm{F}})$ if and only if $\mathbf{A} \in \mathrm{S}_{\mathrm{P}}(\mathcal{K}_{\mathrm{F}})$. $\square$

The remaining part is devoted to a general finite α-embeddability theorem which is necessary for proving the uniform variants of Di Nola's representation Theorem (see [56]). At first we recall some definitions.

Definition 1.9. [44] Let α be an infinite cardinal. A proper filter D of $\wp(I)$ is said to be *α-regular* if there exists a set $E \subseteq D$ such that $|E| = \alpha$ and each $i \in I$ belongs to only finitely many $e \in E$.

Definition 1.10. Let α be an infinite cardinal, let $\mathbf{A} = (A, \mathrm{F})$ be an algebra such that $|A| \leq \alpha$. Let $i_A : A \to \alpha$ be the respective injective mapping. $\mathbf{A}$ satisfies the *general finite α-embeddability* (*finite α-embeddability*) *property for the class* $\mathcal{K}$ of algebras of the same type if for any finite subset $X \subseteq \alpha$, there exist an (finite) algebra $\mathbf{B} \in \mathcal{K}$ and an embedding $\rho : \mathbf{A}|_{i_A^{-1}(X)} \hookrightarrow \mathbf{B}$, i.e., an injective mapping $\rho : i_A^{-1}(X) \to B$ satisfying the property $\rho(f^{\mathbf{A}|_{i_A^{-1}(X)}}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \ldots, \rho(x_n))$ if $f \in \mathrm{F}_n$, $x_1, \ldots, x_n \in A$, $i_A(x_1), \ldots, i_A(x_n) \in X$, and $f^{\mathbf{A}|_{i_A^{-1}(X)}}(x_1, \ldots, x_n)$ is defined.

The following theorem is a modification of Theorem 1.3 for algebras of a bounded cardinality.

Theorem 1.5. *Let α be an infinite cardinal and let $\mathbf{A} = (A, \mathrm{F})$ be an algebra such that $|A| \leq \alpha$. Let $i_A : A \to \alpha$ be the respective injective mapping. Let $\mathcal{K}$ be a class of algebras of the same type. If $\mathbf{A}$ satisfies the general finite α-embeddability property for $\mathcal{K}$ then there is an α-regular ultrafilter U over the set $I = \{X \subseteq \alpha \mid X \text{ is finite}\}$, which does not depend on A, and algebras $\mathbf{A}_X \in \mathcal{K}$, $X \subseteq \alpha$ finite such that A can be embedded into $(\prod_{X \in I} \mathbf{A}_X)/U$.*

Proof. Let $\mathbf{A}$ satisfy the general finite α-embeddability property for $\mathcal{K}$. Then for any $X \in I$ there exist $\mathbf{A}_X \in \mathcal{K}$ and an embedding $\rho_X : \mathbf{A}|_{i_A^{-1}(X)} \hookrightarrow \mathbf{A}_X$. By the axiom of choice we choose a fixed $a_X \in A_X$ for any $X \in I$. Now we define a mapping $\varphi : \mathbf{A} \to \prod_{X \in I} \mathbf{A}_X$ by

$$\varphi(a)(X) = \begin{cases} \rho_X(a) & \text{if } i_A(a) \in X; \\ a_X & \text{otherwise.} \end{cases}$$

Denote further $U(X) = \{Y; Y \in I \text{ and } X \subseteq Y\}$ and $V = \{U(X); X \in I\}$. Then for any $U(X), U(Y) \in V$, the equality $U(X) \cap U(Y) = U(X \cup Y) \in V$ holds and thus $U(X) \cap U(Y) \neq \emptyset$. Consequently, there is an ultrafilter U of $\wp(I)$ such that $V \subseteq U$.

Let us check that U is α-regular. Let us put $E = \{U(\{x\}) \mid x \in \alpha\}$. Evidently, $|E| = \alpha$ and, for any $X \in I$ we have that $X \in U(\{x\})$ iff $x \in X$.

Therefore any $X \in I$ belongs to only finitely many elements of E because X is a finite subset of α.

Hence, we can define a mapping

$$\rho : \mathbf{A} \to (\prod_{X \in I} \mathbf{A}_X)/U$$

such that $\rho(a) = \varphi(a)/U$. In Theorem 1.3 it was proved that defined mapping ρ is an injective homomorphism. $\square$

Chapter 2

MV-algebras

MV-algebras were introduced by C.C. Chang [42] as an algebraization of Łukasiewicz many-valued propositional logic. The main idea of his definition is to present a logic with truth scale $[0,1] \subseteq \mathbb{R}$ where disjunction is represented by the cut addition

$$x \oplus y := \min\{x+y, 1\}$$

and negation is defined as an antitone involution by $\neg x := 1 - x$. This model is called the *standard MV-algebra*. The Łukasiewicz many-valued logic (and thus also MV-algebras) became very useful in applications of fuzzy logics for its simplicity and its naturality.

Recall that by an *MV-algebra* is meant an algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ of a type $\langle 2,1,0\rangle$ satisfying the axioms:

(MV1) $x \oplus y = y \oplus x$,

(MV2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus 1 = 1$, where $1 := \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

It was proved by M. Kolařík (see [107]) that this axiom system is not independent. The axiom (MV1) follows from (MV2)-(MV6).

The order relation $\leq$ can be introduced on any MV-algebra $\mathbf{A}$ by the stipulation

$$x \leq y \text{ if and only if } \neg x \oplus y = 1.$$

Moreover, the ordered set $(A; \leq)$ can be organized into a bounded distributive lattice[a] $(A; \vee, \wedge, 0, 1)$ where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \text{ and } x \wedge y = \neg(\neg x \vee \neg y).$$

In the theory of MV-algebras we use derived operations $\ominus$, $\odot$ and $\to$ defined by $x \ominus y := \neg(\neg x \oplus y)$, $x \odot y := \neg(\neg x \oplus \neg y)$ and $x \to y := \neg x \oplus y$. The last two operations are connected by the adjointness property

$$x \odot y \leq z \text{ if and only if } x \leq y \to z.$$

An MV-algebra is said to be *linearly ordered* (or an *MV-chain*) if its order is linear.

Given a positive integer $n \in \mathbb{N}$, we let $n \times x = x \oplus x \oplus x \cdots \oplus x$, n times, $x^n = x \odot x \odot x \cdots \odot x$, n times, $0x = 0$ and $x^0 = 1$.

In every MV-algebra the following equalities hold:

(D1) $a \oplus \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \oplus x_i)$, $a \oplus \bigwedge_{i \in I} x_i = \bigwedge_{i \in I}(a \oplus x_i)$,

(D2) $a \odot \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \odot x_i)$, $a \odot \bigwedge_{i \in I} x_i = \bigwedge_{i \in I}(a \odot x_i)$,

whenever the left sides are defined.

An element a of an MV-algebra $\mathbf{A}$ is said to be *Boolean* if $a \oplus a = a$. We say that an MV-algebra $\mathbf{A}$ is *Boolean* if every element of $\mathbf{A}$ is Boolean. For an MV-algebra $\mathbf{A}$, the set $B(\mathbf{A})$ of all Boolean elements is a Boolean algebra.

Morphisms of MV-algebras (shortly *MV-morphisms*) are defined as usual, i.e., they are functions which preserve the binary operations $\oplus$ and $\odot$, the unary operation $\neg$ and the constants 0 and 1.

[a] We stress that the operation $\neg$ is not a complement in MV-algebras. More precisely, $\neg$ is a complement operation if and only if the MV-algebra is a Boolean algebra.

Example 2.1. If $\mathbf{B} = (B; \vee, \wedge, \neg, 0, 1)$ is an Boolean algebra then the reduct $(B; \vee, \neg, 0)$ is an MV-algebra.

The following theorem describes the correspondence between MV-algebras and BL-algebras with the double negation law.

Theorem 2.1. *i) If* $\mathbf{A} = (A; \oplus, \neg, 0)$ *is an MV-algebra then the algebra* $(A; \vee, \wedge, \odot, \to, 1, 0)$ *is a BL-algebra satisfying the double negation law* $\neg\neg x = x$.

ii) If $\mathbf{B} = (B; \vee, \wedge, \cdot, \to, 1, 0)$ *is a BL-algebra satisfying double negation law, then* $(B; \oplus, \neg, 0)$*, where* $x \oplus y := \neg(\neg x \cdot \neg y)$ *and* $\neg x := x \to 0$*, is an MV-algebra.*

Thanks to this theorem we can apply Definition 1.6 of filters and Theorem 1.2 for MV-algebras.

Chang's famous results show the importance of the standard MV-algebra for the theory of MV-algebras, see [46] for more details.

Theorem 2.2. *The variety of MV-algebras* $\mathcal{MV}$ *is generated by the standard MV-algebra.*

The theory of MV-algebras is well developed and there are many interesting results and connections with other parts of mathematics. Firstly, the category of MV-algebras is equivalent to the category of commutative ℓ-groups (lattice ordered commutative groups) with strong order unit. Secondly, the variety (or quasivariety) of MV-algebras is generated by the standard MV-algebra. Consequently, the free algebras are subalgebras of direct power $[0, 1]^X$ of the standard MV-algebra and thus MV-algebras are related to some basic geometrical theories (see [119]). Most of the very deep results, though, are dependent on some representation theorems.

2.1 Extensions of Di Nola's Theorem

The representation theory of MV-algebras is based on Chang's representation Theorem [42], McNaughton's Theorem and Di Nola's representation

Theorem [56]. Chang's representation Theorem yields a subdirect representation of all MV-algebras via linearly ordered MV-algebras. McNaughton's Theorem characterizes free MV-algebras as algebras of continuous, piecewise linear functions with integer coefficients on $[0, 1]$. Finally, Di Nola's representation Theorem describes MV-algebras as subalgebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra $[0, 1]$.

The main motivation for our research comes from the fact that although the proofs of both Chang's representation Theorem and McNaughton's Theorem are of algebraic nature, the proof of Di Nola's representation Theorem is based on model-theoretic considerations. We give a simple, purely algebraic, proof of it and its variants based on Farkas' lemma for rationals [78] and the general finite embedding theorem [19]. Similar results for zero-cancellative pomonoids were obtained in [128]. As a surprising by-product we obtain in Theorem 2.5 a new proof of the completeness of the Łukasiewicz axioms [43]. Farkas' lemma or its alternatives were used in the theory of MV-algebras also by Gispert and Mundici (see [91, 46]).

2.1.1 Farkas' lemma

Let us recall the original formulation of Farkas' lemma [78, 140] on rationals.

Theorem 2.3 (Farkas' lemma)**.** *Given a matrix A in $\mathbb{Q}^{m\times n}$ and $\boldsymbol{c}$ a column vector in $\mathbb{Q}^m$, there exists a column vector $\boldsymbol{x} \in \mathbb{Q}^n$, $\boldsymbol{x} \geq \mathbf{0}_n$ such that $A \cdot \boldsymbol{x} = \boldsymbol{c}$ if and only if for all row vectors $\boldsymbol{y} \in \mathbb{Q}^m$, $\boldsymbol{y} \cdot A \geq \mathbf{0}_m$ implies $\boldsymbol{y} \cdot \boldsymbol{c} \geq 0$.*

In what follows, we will use the following equivalent formulation:

Theorem 2.4 (Theorem of alternatives)**.** *Let A be a matrix in $\mathbb{Q}^{m\times n}$ and let $\boldsymbol{b}$ be a column vector in $\mathbb{Q}^m$. The system $A \cdot \boldsymbol{x} \leq \boldsymbol{b}$ has no solution if and only if there exists a row vector $\boldsymbol{\lambda} \in \mathbb{Q}^m$ such that $\boldsymbol{\lambda} \geq \mathbf{0}_m$, $\boldsymbol{\lambda} \cdot A = \mathbf{0}_n$ and $\boldsymbol{\lambda} \cdot \boldsymbol{b} < 0$.*

Remark 2.1. Since the row vector $\boldsymbol{\lambda} \in \mathbb{Q}^m$ from Theorem 2.4 has non-negative rational components $\lambda_i = \frac{p_i}{q_i}$, $p_i \in \mathbb{N}_0$, $q_i \in \mathbb{N}$, we may assume (by taking the least common multiple q of the denominators q_i and multiplying by it the components of $\boldsymbol{\lambda}$) that $\boldsymbol{\lambda} \in \mathbb{Z}^m$.

2.1.2 The Embedding lemma

In this part, we use Farkas' lemma on rationals to prove that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into $\mathbb{Q}\cap[0,1]$ and hence into the finite MV-chain $\mathbf{L}_k \subseteq [0,1]$ for a suitable $k \in \mathbb{N}$.

Lemma 2.1. *Let $\mathbf{M} = (M; \oplus, \neg, 0)$ be a linearly ordered MV-algebra and let $X \subseteq M \setminus \{0\}$ be a finite subset. Then there is a rational-valued map $s : X \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ such that*

1. *$s(0) = 0$, $s(1) = 1$,*
2. *if $x, y, x \oplus y \in X \cup \{0,1\}$ such that $x \leq \neg y$ and $x, y \in X \cup \{0,1\}$ then $s(x \oplus y) = s(x) + s(y)$,*
3. *if $x \in X$ then $s(x) > 0$.*

Proof. We put

$$Y(X) := \{x \oplus y \mid x, y \in X \cup \{0,1\}\}.$$

Thus $Y(X) \subseteq M$ is finite, $X \subseteq Y(X)$, $0,1 \in Y(X)$, $X \oplus X \subseteq Y(X)$. Since $\mathbf{M}$ is a chain, we may assume that $Y(X) = \{y_0 = 0 < y_1 < \cdots < y_n = 1\}$ and put $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in Y(X)^n$. For any $x \in X$, there is an index $1 \leq j_x \leq n$ such that $x = y_{j_x}$. If $x \leq \neg y, x \neq y$ and $x, y \in X$, we denote by $\mathbf{a}^1_{x,y}, \mathbf{a}^2_{x,y} \in \mathbb{Z}^n$ non-negative row vectors such that

$$\mathbf{a}^1_{x,y}(j) = \begin{cases} 1 & \text{if } j = j_x \text{ or } j = j_y; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{a}^2_{x,y}(j) = \begin{cases} 1 & \text{if } j = j_{x\oplus y}; \\ 0 & \text{otherwise.} \end{cases}$$

If $x \leq \neg x$ and $x \in X$, we denote by $\mathbf{a}^1_{x,x}, \mathbf{a}^2_{x,x} \in \mathbb{Z}^n$ non-negative row vectors such that

$$\mathbf{a}^1_{x,x}(j) = \begin{cases} 2 & \text{if } j = j_x; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{a}^2_{x,x}(j) = \begin{cases} 1 & \text{if } j = j_{x\oplus x}; \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $i \in \{1, 2\}$. Let A^i be a matrix consisting of the rows $\mathbf{a}^i_{x,y}$ such that $x \leq \neg y$ and $x, y \in X$. We put $A = A^1 - A^2$ and $m = |\{\{x, y\} \mid x, y \in X, x \leq \neg y\}|$.

Due to Mundici's equivalence our linearly ordered MV-algebra $\mathbf{M}$ is an interval $[0, u]$ in a linearly ordered commutative ℓ-group $\mathbf{G}$ with a strong unit u (Chang's ℓ-group). If $v, w \in M$ are such that $w \leq \neg v$ then the sum $v \oplus w$ coincides with the sum $v + w$ computed in $\mathbf{G}$.

It follows that

$$A^1 \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A^2 \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

hence

$$A \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \mathbf{0}_m,$$

the result being computed in $\mathbf{G}$.

Let E_n be the identity matrix of order n. Let us denote by $(*)$ the following system of linear inequalities with variables $z_1, \ldots, z_n$ over rationals:

$$\begin{pmatrix} -E_n \\ A \\ -A \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \leq \begin{pmatrix} c - \mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix}. \tag{$*$}$$

Then by Farkas' lemma (see Theorem 2.4) for rationals the systems of inequalities $(*)$ does not have a solution in $\mathbb{Q}^n$ if and only if there is a row vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n+2m}) \in \mathbb{Z}^{n+2m}$, $\boldsymbol{\lambda} \geq \mathbf{0}_{n+2m}$ such that

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} -E_n \\ A \\ -A \end{pmatrix} = 0, \ \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} < 0. \tag{$**$}$$

Assume that there exists a vector $\boldsymbol{\lambda} \in \mathbb{Z}^{n+2m}$ satisfying $(**)$. Hence there is an index $1 \leq j_0 \leq n$ such $\lambda_{j_0} > 0$.

We then have (again computing in $\mathbf{G}$) that

$$0 = \boldsymbol{\lambda} \cdot \begin{pmatrix} -E_n \\ A \\ -A \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{y} \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} = -\sum_{j=1}^{n} \lambda_j y_j \qquad (***)$$

(here $\boldsymbol{\lambda}$ is a row vector of order $n+2m$ over non-negative integers, $\begin{pmatrix} -E_n \\ A \\ -A \end{pmatrix}$ is a matrix of type $(n+2m)\times n$ over integers, $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a column vector of order n over elements from $\mathbf{G}$, and $\begin{pmatrix} -\mathbf{y} \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix}$ is a column vector of order $n+2m$ over elements from $\mathbf{G}$).

Since $\lambda_1, \ldots, \lambda_{n+2m}$ are non-negative, λ_{j_0} is positive, and $y_1, \ldots, y_n$ are positive non-zero elements in $\mathbf{G}$, we get that $\sum_{i=1}^{n} \lambda_j y_j$ is a positive non-zero element from $\mathbf{G}$, which contradicts $(***)$.

It follows that system $(*)$ has a rational-valued solution $(q_1, \ldots, q_n)$, and from $(*)$ it clearly follows that the solution is positive (more precisely $\mathbf{q} = (q_1, \ldots, q_n) \geq \mathbf{1}_n$). We define a map $s : X \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ by:

$$s(x) = \begin{cases} \frac{q_{j_x}}{q_n} & \text{if } x \in X; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x = 1. \end{cases}$$

The map s evidently satisfies conditions (1)–(3) of this lemma. □

Remark 2.2. Another possibility how to prove Lemma 2.1 is to use the proof of [48, Theorem 1] with respective restrictions given by the conditions (1)–(3) for a totally ordered abelian group $\mathbf{G}$ corresponding in the Mundici duality to our linearly ordered MV-algebra $\mathbf{M}$.

Lemma 2.2 (**Embedding lemma**). *Let $\mathbf{M} = (M; \oplus, \neg, 0)$ be a linearly ordered MV-algebra and let $X \subseteq M$ be a finite set. Then there exists an embedding $f : \mathbf{X} \hookrightarrow \mathbf{L}_k$, where $\mathbf{X}$ is a partial MV-algebra obtained by the restriction of $\mathbf{M}$ to the set X, and $\mathbf{L}_k \subseteq [0,1]$ is the linearly ordered finite MV-algebra on the set $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$.*

Proof. Let us define a set Y as follows:

$$Y := \{x \ominus y \mid x, y \in X \cup \{0,1\}\} \setminus \{0\}.$$

Moreover, let $s : Y \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ be the respective mapping for the set Y from Lemma 2.1.

Let $f = s|_X$ be the restriction of the mapping s to the set X. We first prove that f is injective. Let $x < y$, $x, y \in X$. Then there is $z \in Y$, $z \neq 0$ such that $x \leq \neg z$ and $x \oplus z = y$. It follows that $0 < s(z)$ and $s(y) = s(x \oplus z) = s(x) + s(z) > s(x)$, i.e., $f(x) = s(x) < s(y) = f(y)$.

Let $f(X) \setminus \{0\} = \{\frac{p_1}{q_1}, \ldots, \frac{p_l}{q_l}\}$ for some $p_1, q_1, \ldots, p_l, q_l \in \mathbb{N}$ and let us denote by k the least common multiple of the denominators $q_1, \ldots, q_l$. Then evidently $f(X) \subseteq \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$.

If $0 \in X$ then by definition of s, we obtain $f(0) = s(0) = 0$. If $x, \neg x \in X$ for some $x \in M$ then clearly $\neg x \leq \neg x$ and using Lemma 2.1, we obtain $s(\neg x) + s(x) = s(\neg x \oplus x) = s(1) = 1$. Hence, $f(\neg x) = s(\neg x) = 1 - s(x) = \neg s(x) = \neg f(x)$.

Finally, let $x, y \in X$ be such that $x \oplus y \in X$. Then

1. If $x \leq \neg y$ then $f(x \oplus y) = s(x \oplus y) = s(x) + s(y) = f(x) \oplus f(y)$.

2. If $\neg y < x$ then $f(x \oplus y) = f(1) = s(1) = 1$. Conversely, $x \ominus \neg y \in Y$ and $x \ominus \neg y = y \ominus \neg x \leq y$ and thus $s(x) = s((x \ominus \neg y) \oplus \neg y) = s(x \ominus \neg y) + s(\neg y) \geq s(\neg y) = 1 - s(y)$. It follows that $1 \geq f(x) \oplus f(y) = \min(1, s(x) + s(y)) \geq \min(1, (1 - s(y)) + s(y)) = 1$.

Taking the above into consideration, f is an injective partial MV-morphism, hence an embedding. $\square$

2.1.3 An elementary proof of the completeness of the Łukasiewicz axioms

In what follows, for the notion of an MV-equation we refer to [46]. Informally, an MV-equation is any equation in the first-order language of MV-algebras. We shall tacitly identify propositional formulas and MV-terms.

A *valuation* e on an MV-algebra $\mathbf{M}$ is an MV-morphism from the free MV-algebra $F_{\mathcal{MV}}(\aleph_0)$ over countably many generators $x_1, \ldots, x_n, \ldots$ (the so-called *term MV-algebra*) to the MV-algebra $\mathbf{M}$. Note that e is uniquely determined by the action on generators $x_1, \ldots, x_n, \ldots$.

The notion of satisfaction (validity) of an MV-equation in an MV-algebra $\mathbf{M}$ is the usual model-theoretic notion of satisfaction (validity).

Chang's Completeness Theorem is stated in the form of [48, Theorem 2.5.3]. This formulation then guarantees the completeness of the Łukasiewicz axioms (see [48, Chapter 4]).

Theorem 2.5 (Completeness Theorem for $[0,1]$). *An equation holds in $[0,1]$ if and only if it holds in every MV-algebra.*

Proof. Suppose that an MV-equation $\tau = \sigma$ in the variables $x_1, \ldots, x_n$ fails in an MV-algebra $\mathbf{A}$. By Chang's representation Theorem it fails in some linearly ordered MV-algebra $\mathbf{M}$. It follows that there is a valuation e on $\mathbf{M}$ such that $e(\tau) \neq e(\sigma)$. Denote by T the set of all subterms of both τ and σ. We put $X = e(T) \subseteq M$. Clearly, X is finite. By Lemma 2.2, there is an embedding $f : \mathbf{X} \hookrightarrow [0,1]$. Let $f \circ e$ be a valuation on $[0,1]$ given by the action

$$x_i \mapsto \begin{cases} f(e(x_i)) & \text{if } i \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

This yields that $f \circ e$ is a valuation on $[0,1]$ such that $(f \circ e)(\tau) \neq (f \circ e)(\sigma)$. □

Remark 2.3. Note that in the main monograph about MV-algebras ([48, Theorem 2.5.3]), one can find a geometric proof of the completeness of the Łukasiewicz axioms relaying on elementary algebra, and also references to other its proofs based on syntactic methods and linear inequalities, on the represention of free ℓ-groups, as well as on techniques from algebraic geometry.

Another way to obtain the completeness of the Łukasiewicz deductive system is via the arguments used in [76] in conjunction with the existence of the algebra $M(\alpha)$, mentioned in Remark 2.5 on page 37, into which every linearly ordered MV-algebra of bounded cardinality α embeds, or by the use of the uniform form of Di Nola's representation Theorem, also granting such an embedding.

The difference between our proof and the already known ones is based on the fact that we first show our Embedding Lemma (using only Chang's ℓ-group and a variant of Farkas' lemma). From this step on, the proofs of several versions of the completeness theorem are straightforward. For example, the proof of [48, Theorem 2.5.3] is based on the correspondence between MV-terms and suitably chosen ℓ-group terms, on the fundamental theorem on torsion-free abelian groups and on a variant of Farkas' lemma as well.

By the same arguments as before we obtain the following two theorems.

Theorem 2.6 (Completeness Theorem for $[0,1]\cap\mathbb{Q}$). *An equation holds in $[0,1]\cap\mathbb{Q}$ if and only if it holds in every MV-algebra.*

Theorem 2.7 (Completeness Theorem for $\mathbf{L}_k$). *An equation holds in $\mathbf{L}_k$ for all $k\in\mathbb{N}$ if and only if it holds in every MV-algebra.*

2.1.4 Extensions of Di Nola's Theorem

In this part, we are going to show Di Nola's representation Theorem and its several variants not only via the standard MV-algebra $[0,1]$ but also via its rational part $\mathbb{Q}\cap[0,1]$ and finite MV-chains. To prove it, we use the Embedding Lemma obtained in the previous section. First, we establish FEP for linearly ordered MV-algebras.

Remark 2.4. Note that part (1) of the following theorem for subdirectly irreducible MV-algebras can be easily deduced from the result that the class of subdirectly irreducible Wajsberg hoops has FEP (see [10, Theorem 3.9]). The well-known part (2) then follows from [10, Lemma 3.7,Theorem 3.9] or from [91, Corollary 9.6].

Theorem 2.8.

1. *The class $\mathcal{LMV}$ of linearly ordered MV-algebras has FEP.*

2. *The class $\mathcal{MV}$ of MV-algebras has FEP.*

Proof. 1) It follows immediately from Lemma 2.2.

2) Let $\mathbf{M} = (M; \oplus, \neg, 0)$ be an MV-algebra and let $X \subseteq M$ be a finite subset. For any $x, y \in X, x \neq y$, there is a prime filter P such that $x/P \neq y/P$. Hence there exists a finite system of prime filters $P_1, \ldots, P_l$, which separates elements from X, i.e., $\mathbf{X} \hookrightarrow \prod_{i=1}^{l}(\mathbf{M}/P_i)$ is an injective mapping. For any $i \in \{1, \ldots, l\}$, by Lemma 2.2, there exists an embedding $f_i : \mathbf{X}/P_i \hookrightarrow \mathbf{L}_{k_i}$. Let k be the least common multiple of $k_1, \ldots, k_l$. Thus, for any $i \in \{1, \ldots, l\}$, there is an embedding $f_i : \mathbf{X}/P_i \hookrightarrow \mathbf{L}_k$. Consequently, there is an embedding $f : \mathbf{X} \hookrightarrow (\mathbf{L}_k)^l$ defined by $f(x)(i) = f_i(x/P_i)$. □

We are now ready to establish the second most important result of our paper - a variant of Di Nola's representation Theorem for finite MV-chains (finite MV-algebras). Our proof, which is based on Theorems 2.8 and 1.3, is entirely algebraic without model-theoretic considerations.

Theorem 2.9.

1. *Any linearly ordered MV-algebra can be embedded into an ultraproduct of finite MV-chains.*

2. *Any MV-algebra can be embedded into a product of ultraproducts of finite MV-chains.*

3. *Any MV-algebra can be embedded into an ultraproduct of finite MV-algebras (which are embeddable into powers of finite MV-chains).*

Proof. (1) It is a direct consequence of Theorem 2.8, (1) and Theorem 1.3.

(2) Any MV-algebra is embeddable into a product of linearly ordered ones. The rest follows by (1).

(3) It is a direct corollary of Theorem 2.8, (2) and Theorem (1). □

The next two theorems cover Di Nola's Representation Theorem and its respective variants both for rationals and reals.

Theorem 2.10.

1. *Any linearly ordered MV-algebra can be embedded into an ultrapower of* $\mathbb{Q} \cap [0, 1]$.
2. *Any MV-algebra can be embedded into a product of ultrapowers of* $\mathbb{Q} \cap [0, 1]$.
3. *Any MV-algebra can be embedded into an ultrapower of a countable power of* $\mathbb{Q} \cap [0, 1]$.
4. *Any MV-algebra can be embedded into an ultraproduct of finite powers of* $\mathbb{Q} \cap [0, 1]$.

Proof. (1)–(4) are corollaries of Theorem 2.9. □

Theorem 2.11.

1. *Any linearly ordered MV-algebra can be embedded into an ultrapower of* $[0, 1]$.
2. *Any MV-algebra can be embedded into a product of ultrapowers of* $[0, 1]$.
3. *Any MV-algebra can be embedded into an ultrapower of a countable power of* $[0, 1]$.
4. *Any MV-algebra can be embedded into an ultraproduct of finite powers of* $[0, 1]$.

Proof. (1)–(4) are corollaries of Theorem 2.9. □

2.1.5 Representation of MV-algebras by regular ultrapowers

In this section we present a uniform version of Di Nola's Theorem for rationals. This enables us to embed all MV-algebras of a cardinality at most α in an algebra of functions from 2^α into a single non-standard ultrapower of the MV-algebra $\mathbb{Q} \cap [0,1]$. Our second goal is to embed all MV-algebras of a cardinality at most α into a single non-standard ultrapower of the MV-algebra $(\mathbb{Q} \cap [0,1])^{\mathbb{N}}$.

Theorem 2.12. *Let α be an infinite cardinal and let $\mathbf{M} = (M; \oplus, \neg, 0)$ be a linearly ordered MV-algebra such that $|M| \leq \alpha$. Let U be the α-regular ultrafilter over the set $I = \{X \subseteq \alpha \mid X \text{ is finite}\}$ from Theorem 1.5 which does not depend on $\mathbf{M}$. Then*

1. *$\mathbf{M}$ can be embedded into an ultraproduct of finite MV-chains via the α-regular ultrafilter U.*

2. *$\mathbf{M}$ can be embedded into the ultrapower $(\prod_{X \in I} \mathbb{Q} \cap [0,1])/U$.*

3. *$\mathbf{M}$ can be embedded into the ultrapower $(\prod_{X \in I}[0,1])/U$.*

Proof. (1)–(3) are corollaries of Theorem 2.8(1) and Theorem 1.5. □

Remark 2.5. Recall that Wojciechowski has shown in [145, Theorem 2.4] that given a cardinal number α, there exists a totally ordered MV-algebra $\mathbf{M}(\alpha)$ such that every totally ordered MV-algebra of cardinality at most α embeds into $\mathbf{M}(\alpha)$. In the proof he used the Hahn o-group $V(\Lambda(\alpha), \mathbb{Q})$ (see [96]) on some suitable totally ordered set $\Lambda(\alpha)$. Our approach is based entirely on Theorem 1.5 and on the consequences of the Embedding lemma.

Theorem 2.13. *Let α be an infinite cardinal and let $\mathbf{M} = (M; \oplus, \neg, 0)$ be an MV-algebra such that $|M| \leq \alpha$. Let U be the α-regular ultrafilter over the set $I = \{X \subseteq \alpha \mid X \text{ is finite}\}$ from Theorem 1.5 which does not depend on $\mathbf{M}$. Then*

1. *$\mathbf{M}$ can be embedded into an MV-algebra of functions from 2^α to the ultrapower $(\prod_{X\in I}\mathbb{Q}\cap[0,1])/U$.*

2. *$\mathbf{M}$ can be embedded into an MV-algebra of functions from 2^α to the ultrapower $(\prod_{X\in I}[0,1])/U$.*

Proof. (1) Let $i_M : M \to \alpha$ be an injective mapping. Let $PFilt(\mathbf{M})$ be the set of all prime filters of $\mathbf{M}$ (which is evidently non-empty) and let $F_0 \in PFilt(\mathbf{M})$. By Chang's representation Theorem, we have an embedding

$$f : \mathbf{M} \hookrightarrow \prod_{F\in PFilt(\mathbf{M})} \mathbf{M}/F.$$

Moreover, we have an injective mapping $e_M : PFilt(\mathbf{M}) \to 2^\alpha$ given by $F \mapsto \{i_M(x) \mid x \in F\} \subseteq \alpha$. For any $F \in PFilt(\mathbf{M})$, Theorem 2.12 provides an embedding

$$g_F : \mathbf{M}/F \hookrightarrow (\prod_{X\in I}\mathbb{Q}\cap[0,1])/U.$$

This yields an embedding

$$g : \prod_{F\in PFilt(\mathbf{M})} \mathbf{M}/F \hookrightarrow \left((\prod_{X\in I}\mathbb{Q}\cap[0,1])/U\right)^{2^\alpha}$$

given as follows:

$$g((x_F)_{F\in PFilt(\mathbf{M})})(B) = \begin{cases} g_F(x_F) & \text{if } e_M(F) = B \\ g_{F_0}(x_{F_0}) & \text{otherwise.} \end{cases}$$

The composition of $g \circ f$ gives us the required embedding.
(2) The proof follows the considerations of (1). □

Going the other way around, we have the following.

Theorem 2.14. *Let α be an infinite cardinal and let $\mathbf{M} = (M; \oplus, \neg, 0)$ be a MV-algebra such that $|M| \leq \alpha$. Let U be the α-regular ultrafilter over the set $I = \{X \subseteq \alpha \mid X \text{ is finite}\}$ from Theorem 1.5 which does not depend on $\mathbf{M}$. Then*

1. **M** *can be embedded into the ultrapower* $\left(\prod_{X\in I}(\mathbb{Q}\cap[0,1])^{\mathbb{N}}\right)/U$.

2. **M** *can be embedded into the ultrapower* $\left(\prod_{X\in I}[0,1]^{\mathbb{N}}\right)/U$.

Proof. (1) Let $i_M : M \to \alpha$ be an injective mapping and let $X \subseteq M$ be a finite subset. Using the same notation and reasonings as in the proof of Theorem 2.8(2) we have an embedding

$$f : \mathbf{X} \hookrightarrow (\mathbf{L}_k)^l.$$

Moreover, we have also an embedding

$$g : (\mathbf{L}_k)^l \hookrightarrow (\mathbb{Q}\cap[0,1])^{\mathbb{N}}$$

given by:

$$g((x_k)_{k=1}^l)(n) = \begin{cases} x_k & \text{if } k = n; \\ x_1 & \text{otherwise.} \end{cases}$$

The composition $\rho_X = g \circ f$ yields an embedding

$$\rho_X : \mathbf{X} \hookrightarrow (\mathbb{Q}\cap[0,1])^{\mathbb{N}}.$$

The remaining part now follows from Theorem 1.5.
(2) The proof follows the considerations of (1). □

Remark 2.6. In the paper [104] Keisler proved that given an infinite cardinal α, if U is regular and κ is an infinite cardinal such that I is taken as in Theorem 2.13, then $\mathrm{card}((\prod_{X\in I}\kappa)/U) = \mathrm{card}(\kappa^\alpha)$.

Hence we immediately get that

$$\mathrm{card}\left(\left((\prod_{X\in I}\mathbb{Q}\cap[0,1])/U\right)^{2^\alpha}\right) = \aleph_0^{2^\alpha}$$

and

$$\mathrm{card}\left(\left(\prod_{X\in I}(\mathbb{Q}\cap[0,1])^{\mathbb{N}}\right)/U\right) = 2^\alpha.$$

It follows that given an infinite cardinal α, there exists a single MV-algebra of the cardinality 2^α, in which every MV-algebra of cardinality at most α embeds.

Second, by the same arguments as in [59, Section 4], for every infinite cardinal α, there is an iterated ultrapower (see [44, Section 6.5]) of $(\mathbb{Q} \cap [0, 1])^{\mathbb{N}}$, definable in α, in which every MV-algebra of cardinality at most α embeds.

2.2 Tense MV-algebras

Propositional logic usually does not incorporate the dimension of time. To obtain to so-called *tense logic*, the propositional calculus is enriched by adding new unary operators G and H (and new derived operators $F := \neg G \neg$ and $P := \neg H \neg$, where $\neg$ denotes the classical negation connective) which are called *tense operators*. The operator G usually expresses the quantifier 'it will still be the case that' and H expresses 'it has always been the case that'. Hence, F and P are in fact tense existential quantifiers.

If T is a non-void set and R a binary relation on T, the couple (T, R) is called a *(time) frame*. T means a *time scale* and R is the so-called *time preference*, i.e., if tRs for $s, t \in T$ then t is assumed to be 'before s' and s is 'after t'. Let us note that R need not be reflexive, nor antisymmetric, nor transitive but, in well-founded applications in quantum mechanics, reflexivity and transitivity is usually assumed.

For a given logical formula ϕ of our propositional logic and for $t \in T$ we say that $G(\phi)(t)$ is valid if $\phi(s)$ is valid for any $s \in T$ with tRs. Analogously, $H(\phi)(t)$ is valid if $\phi(s)$ is valid for any $s \in T$ with sRt. Thus $F(\phi)(t)$ is valid if there exists $s \in T$ with tRs and $\phi(s)$ is valid and analogously $P(\phi)(t)$ is valid if there exists $s \in T$ with sRt and $\phi(s)$ is valid in the propositional logic.

Study of tense operators was originated in 1980's, see e.g. a compendium [26]. Recall that for a classical propositional calculus represented by means of a Boolean algebra $\mathbf{B} = (B; \vee, \wedge, \neg, 0, 1)$, tense operators were axiomatized in [26] by the following axioms:

(B1) $G(1) = 1$, $H(1) = 1$,

(B2) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,

(B3) $\neg G \neg H(x) \leq x$, $\neg H \neg G(x) \leq x$.

For Boolean algebras, the axiom (B3) is equivalent to

(B3') $G(x) \vee y = x \vee H(y)$.

To introduce tense operators in non-classical logics, some more axioms must be added on G and H to express connections with additional operations or logical connectives. For example, for intuitionistic logic (corresponding to Heyting algebras) it was done in [30], for algebras of logic of quantum mechanics see [37] and [40], for so called basic algebras it was done in [23], for other interesting algebras the reader is referred to [79], [80] and [110].

Among algebras connected with many-valued logic, let us mention MV-algebras and Łukasiewicz-Moisil algebras. Tense operators for the previous cases were introduced and studied in [45] and [54]. Contrary to Boolean algebras where the representation problem through a time frame is solved completely, authors in [54] only mention that this problem for MV-algebras was not treated. Hence, our main goal is to find a suitable time frame for given tense operators on a semisimple MV-algebra, i.e., to solve the representation problem for semisimple MV-algebras.

This problem was solved by Botur[b] for such tense MV-algebras that the tense operators G and H preserve all finite powers of the operations $\oplus$ and $\odot$. Paseka generalized his results replacing original term t_q (see [127]) constructed for any rational q by the Teheux's term (see [127]). Here, we present more general concept of used ideas for obtaining stronger results. The main representation theorem for semisimple tense MV-algebras and Paseka's results are corollaries of this. The key and new notion is the concept of a semi-state on an MV-algebra. The advantage is that these semi-states reflect an important property of the logic of quantum mechanics, namely the so-called Jauch-Piron property (see [130]) saying that if the probability of propositions A and B being true is zero then there exists a proposition C covering both A and B. Therefore we may expect applications of our method also in the realm of quantum logic.

Definition 2.1. Let be given an MV-algebra $\mathbf{A} = (A; \oplus, \neg, 0)$. We say that $(\mathbf{A}, G, H)$ is a *tense MV-algebra* and G and H are *tense operators* if G and H are a unary operations on A satisfying:

[b]It is not published.

(i) $G(1) = H(1) = 1$,

(ii) $G(x) \odot G(y) \leq G(x \odot y)$, $H(x) \odot H(y) \leq H(x \odot y)$,

(iii) $G(x) \oplus G(y) \leq G(x \oplus y)$, $H(x) \oplus H(y) \leq H(x \oplus y)$,

(iv) $G(x) \odot G(x) = G(x \odot x)$, $H(x) \odot H(x) = H(x \odot x)$,

(v) $G(x) \oplus G(x) = G(x \oplus x)$, $H(x) \oplus H(x) = H(x \oplus x)$,

(vi) $\neg G \neg H(x) \leq x$, $\neg H \neg G(x) \leq x$.

Applying the axioms (i) and (ii), we get immediately monotonicity of the operators G and H. Thus, if $x \leq y$ for any $x, y \in A$ then $G(x) \leq G(y)$ and $H(x) \leq H(y)$.

We note that the original definition of tense MV-algebras [54, Proposition 5.1, Remark 5.1] uses alternative inequalities

(ii') $G(x \to y) \leq G(x) \to G(y)$, $H(x \to y) \leq H(x) \to H(y)$,

(vi') $x \leq G \neg H \neg x$, $x \leq H \neg G \neg x$.

Monotonicity of the operators G and H and adjointness property give equivalence of (ii) and (ii'). Using double negation law and antitonicity of the negation we obtain equivalence of (vi) and (vi').

The following theorem describes the most important construction of tense MV-algebras.

Theorem 2.15. [54] *Let* $\mathbf{A}$ *be a linearly ordered complete MV-algebra and let* T *be any set with a binary relation* $\rho \subseteq T^2$*. Then* $(\mathbf{A}^T, G^*, H^*)$ *where operations* G^* *and* H^* *are calculated point-wise*

$$G^*(x)(i) := \bigwedge_{i \rho j} x(j) \quad \text{and} \quad H^*(x)(i) := \bigwedge_{j \rho i} x(j)$$

is a tense MV-algebra. In this case we say that the tense MV-algebra $(\mathbf{A}^T, G^*, H^*)$ *is induced by the frame* (T, ρ)*.*

We will prove that any couple of tense operators on any semisimple MV-algebra can be embedded into $([0,1]^T, G^*, H^*)$ where G^* and H^* are tense operators induced by some time frame (T, ρ). For related results on more general operators on any semisimple MV-algebra see [127].

2.2.1 Dyadic numbers and MV-terms

The content of this part summarizes some folklore results of certain MV-terms from [142] and [125]. The techniques described (in a new form) here have been used already in [28] and later, e.g., in [46], [60], [99] and [141].

Since we could not find a proper reference (a journal or a book) on this subject (except [99], [125] and [141] which do not fully cover our needs) and our main result is based on them we review them as follows.

Definition 2.2. [142] The set $\mathbb{D}$ of dyadic numbers is the set of the rational numbers that can be written as a finite sum of integer powers of 2. If a is a number of $[0,1]$, a dyadic decomposition of a is a sequence $a^* = (a_i)_{i\in\mathbb{N}}$ of elements of $\{0,1\}$ such that $a = \sum_{i=1}^{\infty} a_i 2^{-i}$. We denote by a_i^* the i^{th} element of any sequence (of length greater than i) a^*. If a is a dyadic number of $[0,1]$, then a admits a unique finite dyadic decomposition, called the *dyadic decomposition of* a. If a^* is a dyadic decomposition of a real a and if k is a positive integer then we denote by $\ulcorner a^* \urcorner_k$ the finite sequence $(a_1,\dots,a_k)$ defined by the first k elements of a^* and by $\llcorner a^* \lrcorner_k$ the dyadic number $\sum_{i=1}^{k} a_i 2^{-i}$. We denote by $f_0(x)$ and $f_1(x)$ the terms $x \oplus x$ and $x \odot x$ respectively, and by $T_{\mathbb{D}}$ the clone generated by $f_0(x)$ and $f_1(x)$. Here, a *clone* is a set C of finitary operations on a set A such that

- C contains all the projections $\pi_k^n : A^n \to A$, $1 \le k \le n$, defined by:

$$\pi_k^n(x_1,\dots,x_n) = x_k,$$

- C is closed under (finitary multiple) composition: if $f, g_1, \dots, g_m$ are members of C such that f is an m-ary operation, and g_j is an n-ary operation for every j, then the n-ary operation $h(x_1,\dots,x_n) := f(g_1(x_1,\dots,x_n),\dots,g_m(x_1,\dots,x_n))$ is in C.

We also denote by $g_{.}$ the mapping between the set of finite sequences of elements of $\{0,1\}$ (and thus of dyadic numbers in $[0,1]$) and $T_{\mathbb{D}}$ defined by:

$$g_{(a_1,\dots,a_k)} = f_{a_k} \circ \cdots \circ f_{a_1}$$

for any finite sequence $(a_1,\dots,a_k)$ of elements of $\{0,1\}$. If $a = \sum_{i=1}^{k} a_i 2^{-i}$, we sometimes write g_a instead of $g_{(a_1,\dots,a_k)}$.

We also denote, for a dyadic number $a \in \mathbb{D} \cap [0,1]$ and a positive integer $k \in \mathbb{N}$ such that $2^{-k} \leq 1-a$, by $l(a,k) : [a, a+2^{-k}] \to [0,1]$ a linear function defined as follows $l(a,k)(x) = 2^k(x-a)$ for all $x \in [a, a+2^{-k}]$.

Lemma 2.1. [142, Lemma 1.14] *If $a^* = (a_i)_{i\in\mathbb{N}}$ and $x^* = (x_i)_{i\in\mathbb{N}}$ are dyadic decompositions of two elements of $a, x \in [0,1]$, then, for any positive integer $k \in \mathbb{N}$,*

$$g_{\ulcorner a^* \urcorner_k}(x) = \begin{cases} 1 & \textit{if } x > \sum_{i=1}^{k} a_i 2^{-i} + 2^{-k} \\ 0 & \textit{if } x < \sum_{i=1}^{k} a_i 2^{-i} \\ l(\llcorner a^* \lrcorner_k, k)(x) = \sum_{i=1}^{\infty} x_{i+k} 2^{-i} & \textit{otherwise.} \end{cases}$$

Let us define, for any $m \in \mathbb{N}$ a term $\mu_m(x) \in T_{\mathbb{D}}$. First, if $m=1$ we put $\mu_1(x) = f_1(x)$. Second, assume that $\mu_m(x)$ is defined. Then we put $\mu_{m+1}(x) = \mu_1(\mu_m(x))$. It follows that, evaluating $\mu_m(x)$ for $x = \sum_{i=1}^{\infty} x_i 2^{-i}$ in the standard MV-algebra $[0,1]$, we obtain by Lemma 2.1

$$\mu_m(x) = g_{\underbrace{(0,\dots,0)}_{m-\text{times}}}(x) = \begin{cases} 1 & \text{if } x > 2^{-m} \\ \sum_{i=1}^{\infty} x_{i+m} 2^{-i} & \text{otherwise.} \end{cases}$$

Note that for any finite sequence $(a_1,\dots,a_k)$ of elements of $\{0,1\}$ such that $a_k = 0$ we have that $g_{(a_1,\dots,a_k)} = g_{(a_1,\dots,a_{k-1})} \oplus g_{(a_1,\dots,a_{k-1})}$ and clearly any dyadic number a corresponds to such a sequence $(a_1,\dots,a_k)$.

As an immediate consequence, we get

Corollary 2.1. [142, Corollary 1.15 (1)] *Let us have the standard MV-algebra $[0,1]$, $x \in [0,1]$ and $r \in (0,1) \cap \mathbb{D}$. Then there is a term t_r in $T_{\mathbb{D}}$ such that*

$$t_r(x) = 1 \quad \textit{if and only if} \quad r \leq x.$$

Proof. Let $r = \sum_{i=1}^{k} r_i 2^{-i}$ with $r_k = 1$. Then we put $t_r = g_{(r_1,\dots,r_{k-1},0)}$. $\square$

2.2.2 Filters, ultrafilters and the term t_r

The aim of this part is to show that any filter F in an MV-algebra $\mathbf{A}$ which does not contain the element $t_r(x)$ for some dyadic number $r \in (0,1) \cap \mathbb{D}$ and an element $x \in A$ can be extended to an ultrafilter U containing F such that $t_r(x) \notin U$.

A *filter* of an MV-algebra $\mathbf{A}$ is a subset $F \subseteq A$ satisfying:
(F1) $1 \in F$
(F2) $x \in F,\ y \in A,\ x \leq y \ \Rightarrow\ y \in F$
(F3) $x, y \in F \ \Rightarrow\ x \odot y \in F$.

A filter is said to be *proper* if $0 \notin F$. Note that there is a one-to-one correspondence between filters and congruences on MV-algebras. A filter Q is *prime* if it satisfies the following conditions:
(P1) $0 \notin Q$.
(P2) For each x, y in A such that $x \vee y \in Q$, either $x \in Q$ or $y \in Q$.

In this case the corresponding factor MV-algebra $\mathbf{A}/Q$ is linear.

A filter U is *maximal* (and in this case it will be also called an ultrafilter) if $0 \notin U$ and for any other filter F of $\mathbf{A}$ such that $U \subseteq F$, then either $F = A$ or $F = U$. There is a one-to-one correspondence between ultrafilters and MV-morphisms from $\mathbf{A}$ into $[0,1]$ (extremal states). For any ultrafilter U of $\mathbf{A}$ we identify the class x/U with its image in the standard algebra and thus with its image in interval $[0,1]$ of real numbers. Recall that a *state* on $\mathbf{A}$ is a map $s : A \to [0,1]$ satisfying $s(x \oplus y) = s(x) + s(y)$ for all $x, y \in A$ such that $x \odot y = 0$ and $s(1) = 1$.

In what follows we work mostly with MV-morphisms into $[0,1]$ instead of ultrafilters.

Lemma 2.2. *Let $\mathbf{A}$ be a linearly ordered MV-algebra, $s : \mathbf{A} \to [0,1]$ an MV-morphism, $x \in A$ such that $s(x) = 1$. Then $x \oplus x = 1$.*

Proof. Assume that $x \leq \neg x$. Then $1 = s(x) \odot s(x) = s(x \odot x) \leq s(x \odot \neg x) = s(0) = 0$ which is absurd. Therefore $\neg x < x$ and we have that $x \oplus x \geq x \oplus \neg x = 1$. □

Proposition 2.1. *Let* $\mathbf{A}$ *be a linearly ordered MV-algebra,* $s : \mathbf{A} \to [0,1]$ *an MV-morphism,* $x \in A$*. Then* $s(x) = 1$ *iff* $t_r(x) = 1$ *for all* $r \in (0,1) \cap \mathbb{D}$*.*

Equivalently, $s(x) < 1$ *iff there is a dyadic number* $r \in (0,1) \cap \mathbb{D}$ *such that* $t_r(x) \neq 1$*. In this case,* $s(x) < r$*.*

Proof. In what follows we may assume that $x \neq 0$ since $s(0) = 0$ and $t_r(0) = 0$ for all $r \in (0,1) \cap \mathbb{D}$. Note first that $s(t_r(x)) = t_r(s(x))$ since s is an MV-morphism. Then $s(x) = 1$ iff $r \leq s(x)$ for all $r \in (0,1) \cap \mathbb{D}$ iff $t_r(s(x)) = 1$ for all $r \in (0,1) \cap \mathbb{D}$ iff $s(t_r(x)) = 1$ for all $r \in (0,1) \cap \mathbb{D}$.

Assume now that $t_r(x) = 1$ for all $r \in (0,1) \cap \mathbb{D}$. Then evidently $s(t_r(x)) = 1$ for all $r \in (0,1) \cap \mathbb{D}$ and by the above considerations we have that $s(x) = 1$.

Conversely, let $s(x) = 1$ and $r \in (0,1) \cap \mathbb{D}$. Then $t_r(x) = t(x) \oplus t(x)$ such that $t(x)$ is some term from the clone $T_{\mathbb{D}}$ constructed entirely from the operations $(-) \oplus (-)$ and $(-) \odot (-)$. Therefore $s(t(x)) = 1$. By Lemma 2.2 we get that $t_r(x) = t(x) \oplus t(x) = 1$.

□

Proposition 2.2. *Let* $\mathbf{A}$ *be an MV-algebra,* $x \in A$ *and* F *be any filter of* $\mathbf{A}$*. Then there is an MV-morphism* $s : \mathbf{A} \to [0,1]$ *such that* $s(F) \subseteq \{1\}$ *and* $s(x) < 1$ *if and only if there is a dyadic number* $r \in (0,1) \cap \mathbb{D}$ *such that* $t_r(x) \notin F$*.*

Proof. Let $\mathbf{A}$ be an MV-algebra and let F be any filter of $\mathbf{A}$. Assume first that there is an MV-morphism $s : \mathbf{A} \to [0,1]$ such that $s(F) \subseteq \{1\}$ and $s(x) < 1$. Then there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $s(x) < r < 1$. By Corollary 2.1 we get that $s(t_r(x)) = t_r(s(x)) \neq 1$. Hence $t_r(x) \notin F$.

Now, let there be a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t_r(x) \notin F$. Then there is a filter K of $\mathbf{A}$, $F \subseteq K$, $t_r(x) \notin K$ such that K is maximal with this property. Evidently, K is a prime filter of $\mathbf{A}$. Hence the factor algebra $\mathbf{A}/K$ is linearly ordered and we have a surjective MV-morphism $g : \mathbf{A} \to \mathbf{A}/K$, $g(K) \subseteq \{1\}$. Let us denote by U_K the maximal filter of $\mathbf{A}/K$ and by $s_K : \mathbf{A}/K \to [0,1]$ the corresponding MV-morphism. Because $t_r(x) \notin K$ we get that $t_r(g(x)) = g(t_r(x)) \neq 1$.

It follows from Proposition 2.1 that $s_K(t_r(g(x))) < r < 1$. This yields that $s_K(g(t_r(x))) < r < 1$. Let us put $s = s_K \circ g$. Then $s : \mathbf{A} \to [0,1]$ is an MV-morphism, $s(t_r(x)) < r < 1$. Evidently $s(x) < r < 1$ otherwise we would have also $1 = s(t_r(x)) < 1$, a contradiction. Clearly, $s(F) \subseteq s(K) = s_K(g(K)) \subseteq s_K(\{1\}) = \{1\}$. □

Remark 2.7. Recall that in [60] the authors introduced, for each $k \in \mathbb{N}$, $j \in \{0, \dots, k-1\}$, the McNaughton function $s_{j,k} : [0,1] \to [0,1]$ given as follows

$$s_{j,k}(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{j}{k}, \\ -j + kx & \text{if } \frac{j}{k} < x \le \frac{j+1}{k}, \\ 1 & \text{if } \frac{j+1}{k} < x \le 1. \end{cases}$$

In particular, for a dyadic number $a = \sum_{i=1}^{k} a_i 2^{-i} \in [0,1)$, we have that $g_a = s_{j,2^k}$; here $j = a \cdot 2^k < 2^k$.

Moreover, $s_{j,k} \circ s_{l,m} = s_{kl+j,km}$ and $s_{0,k}(x) = k \times x$ and $s_{k-1,k}(x) = x^k$ for all x in the standard MV-algebra $[0,1]$. Similarly as above, one can show, as was first mentioned by Botur, that, for all $r \in (0,1] \cap \mathbb{Q}$, there is a term $\hat{t}_r$ obtained as a composition of terms of the form $k \times x$ and x^l such that

$$\hat{t}_r(x) = 1 \quad \text{if and only if} \quad r \le x.$$

More precisely, we put $\hat{t}_1(x) = x$ and if $r = \frac{p}{q}$, $p, q \in \mathbb{N}$ then we put $\hat{t}_r(x) = \hat{t}_{\frac{q-d}{q}}(c \times (x)^a)$, where $a, b, c, d \in \mathbb{N}$ are such that

$$\begin{aligned} q &= a(q-p) + b & \text{where } 0 < b \le q - p \\ q &= cb + d & \text{where } 0 \le d < b. \end{aligned}$$

Evidently, $\frac{q-d}{q} > \frac{q-b}{q} \ge \frac{q-(q-p)}{q} = \frac{p}{q}$, i.e., the procedure stops after finitely many steps and $\hat{t}_{\frac{q-1}{q}} = (q \times (x)^{q-1}) = s_{q(q-1),q^2}$.

Corollary 2.2. *Let $\mathbf{A}$ be an MV-algebra, $x \in M$ and F be any filter of $\mathbf{A}$ such that $t_r(x) \notin F$ for some dyadic number $r \in (0,1) \cap \mathbb{D}$. Then there is an MV-morphism $s : \mathbf{A} \to [0,1]$ such that $s(F) \subseteq \{1\}$ and $s(x) < r < 1$.*

2.2.3 Semi-states on MV-algebras

In this section we characterize arbitrary meets of MV-morphism into a unit interval as so-called semi-states.

Definition 2.1. *Let* $\mathbf{A}$ *be an MV-algebra. A map* $s : \mathbf{A} \to [0,1]$ *is called*

1. a semi-state on $\mathbf{A}$ *if*

   (i) $s(1) = 1$,

   (ii) $x \leq y$ *implies* $s(x) \leq s(y)$,

   (iii) $s(x) = 1$ *and* $s(y) = 1$ *implies* $s(x \odot y) = 1$,

   (iv) $s(x) \odot s(x) = s(x \odot x)$,

   (v) $s(x) \oplus s(x) = s(x \oplus x)$.

2. a strong semi-state on $\mathbf{A}$ *if it is a semi-state such that*

   (vi) $s(x) \odot s(y) \leq s(x \odot y)$,

   (vii) $s(x) \oplus s(y) \leq s(x \oplus y)$,

   (viii) $s(x \wedge y) = s(x) \wedge s(y)$,

   (ix) $s(x^n) = s(x)^n$ *for all* $n \in \mathbb{N}$,

   (x) $n \times s(x) = s(n \times x)$ *for all* $n \in \mathbb{N}$,

Note that any MV-morphism into a unit interval is a strong semi-state.

Lemma 2.3. *Let* $\mathbf{A}$ *be an MV-algebra,* $s : \mathbf{A} \to [0,1]$ *a semi-state on* $\mathbf{A}$*. Then* s *has the Jauch-Piron property, i.e., for all* $x, y \in A$*,* $s(x) = 1$ *and* $s(y) = 1$ *implies there is* $z \in A, z \leq x$ *and* $z \leq y$ *such that* $s(z) = 1$.

Proof. Let $x, y \in A$ such that $s(x) = 1$ and $s(y) = 1$. From (iii) we get that $z = x \odot y \leq x, y$ satisfies $s(z) = 1$. $\square$

The preceding Jauch-Piron property appears at many places in the axiomatics of quantum systems (see, e.g., [101], [130]). It is well known that any state on an MV-algebra has also the Jauch-Piron property.

Lemma 2.4. *Let* $\mathbf{A}$ *be an MV-algebra,* S *a non-empty set of semi-states (strong semi-states) on* $\mathbf{A}$*. Then the point-wise meet* $t = \bigwedge S : \mathbf{A} \to [0,1]$ *is a semi-state (strong semi-state) on* $\mathbf{A}$*.*

Proof. Let us check the conditions (i)-(vi) from Definition 2.1.

(i): Clearly, $t(1) = \bigwedge\{s(1) \mid s \in S\} = \bigwedge\{1 \mid s \in S\} = 1$.

(ii): Assume $x \leq y$. Then $t(x) = \bigwedge\{s(x) \mid s \in S\} \leq \bigwedge\{s(y) \mid s \in S\} = t(y)$.

(iii): Let $t(x) = \bigwedge\{s(x) \mid s \in S\} = 1$ and $t(y) = \bigwedge\{s(y) \mid s \in S\} = 1$. It follows, that, for all $s \in S$, $s(x) = 1 = s(y)$. Hence also $s(x \odot y) = 1$. This yields that $t(x \odot y) = \bigwedge\{s(x \odot y) \mid s \in S\} = 1$.

(iv, v): Since $[0,1]$ is linearly ordered we have (by taking in the respective part of the proof either the minimum of $s_1(x)$ and $s_2(x)$ or the maximum of $s_1(x)$ and $s_2(x)$)

$$\begin{aligned} t(x) \odot t(x) &= \bigwedge\{s_1(x) \mid s_1 \in S\} \odot \bigwedge\{s_2(x) \mid s_2 \in S\} \\ &= \bigwedge\{s_1(x) \odot s_2(x) \mid s_1, s_2 \in S\} \\ &\geq \bigwedge\{s(x) \odot s(x) \mid s \in S\} \\ &= \bigwedge\{s(x \odot x) \mid s \in S\} = t(x \odot x), \end{aligned}$$

$$\begin{aligned} t(x) \odot t(x) &= \bigwedge\{s_1(x) \odot s_2(x) \mid s_1, s_2 \in S\} \\ &\leq \bigwedge\{s(x) \odot s(x) \mid s \in S\} \\ &= \bigwedge\{s(x \odot x) \mid s \in S\} = t(x \odot x), \end{aligned}$$

$$\begin{aligned} t(x) \oplus t(x) &= \bigwedge\{s_1(x) \mid s_1 \in S\} \oplus \bigwedge\{s_2(x) \mid s_2 \in S\} \\ &= \bigwedge\{s_1(x) \oplus s_2(x) \mid s_1, s_2 \in S\} \\ &\geq \bigwedge\{s(x) \oplus s(x) \mid s \in S\} \\ &= \bigwedge\{s(x \oplus x) \mid s \in S\} = t(x \oplus x), \end{aligned}$$

and

$$\begin{aligned} t(x) \oplus t(x) &= \bigwedge\{s_1(x) \oplus s_2(x) \mid s_1, s_2 \in S\} \\ &\leq \bigwedge\{s(x) \oplus s(x) \mid s \in S\} \\ &= \bigwedge\{s(x \oplus x) \mid s \in S\} = t(x \oplus x). \end{aligned}$$

(vi): Let us compute the following

$$\begin{array}{rcl} t(x)\odot t(y) & = & \bigwedge\{s_1(x)\mid s_1\in S\}\odot\bigwedge\{s_2(y)\mid s_2\in S\}\\ & = & \bigwedge\{s_1(x)\odot s_2(y)\mid s_1,s_2\in S\}\\ & \leq & \bigwedge\{s(x)\odot s(y)\mid s\in S\}\\ & \leq & \bigwedge\{s(x\odot y)\mid s\in S\}=t(x\odot y). \end{array}$$

(vii): Applying the same considerations as in (vi) we have

$$\begin{array}{rcl} t(x)\oplus t(y) & = & \bigwedge\{s_1(x)\mid s_1\in S\}\oplus\bigwedge\{s_2(y)\mid s_2\in S\}\\ & = & \bigwedge\{s_1(x)\oplus s_2(y)\mid s_1,s_2\in S\}\\ & \leq & \bigwedge\{s(x)\oplus s(y)\mid s\in S\}\\ & \leq & \bigwedge\{s(x\oplus y)\mid s\in S\}=t(x\oplus y). \end{array}$$

(viii): Similarly,

$$\begin{array}{rcl} t(x\wedge y) & = & \bigwedge\{s(x\wedge y)\mid s\in S\}=\bigwedge\{s(x)\wedge s(y)\mid s\in S\}\\ & = & \bigwedge\{s(x)\mid s\in S\}\wedge\bigwedge\{s(y)\mid s\in S\}=t(x)\wedge t(y). \end{array}$$

(ix): Assume that $x\in A$ and $n\in\mathbb{N}$. We have, repeatedly using the equality (D2) from the Introduction, that

$$\begin{array}{rcl} (\bigwedge\{s(x);s\in S\})^n & = & \bigwedge\{s_1(x)\odot s_2(x)\odot\cdots\odot s_n(x)\mid s_1,\ldots s_n\in S\}\\ & \leq & \bigwedge\{s(x)\odot s(x)\odot\cdots\odot s(x)\mid s_1,\ldots s_n\in S,\\ & & \quad s\in\{s_1,\ldots,s_n\},s(x)=\max\{s_1(x),\ldots s_n(x)\}\}\\ & = & \bigwedge\{s(x)^n\mid s\in S\}=\bigwedge\{s(x^n)\mid s\in S\} \end{array}$$

and similarly

$$\begin{array}{rcl} (\bigwedge\{s(x);s\in S\})^n & = & \bigwedge\{s_1(x)\odot s_2(x)\odot\cdots\odot s_n(x)\mid s_1,\ldots s_n\in S\}\\ & \geq & \bigwedge\{s(x)\odot s(x)\odot\cdots\odot s(x)\mid s_1,\ldots s_n\in S,\\ & & \quad s\in\{s_1,\ldots,s_n\},s(x)=\min\{s_1(x),\ldots s_n(x)\}\}\\ & = & \bigwedge\{s(x)^n\mid s\in S\}=\bigwedge\{s(x^n)\mid s\in S\} \end{array}$$

Thus $(\bigwedge\{s(x);s\in S\})^n=\bigwedge\{s(x^n)\mid s\in S\}$.

(x): It follows by the same considerations as for (ix) applied to $\oplus$ and repeatedly using the equality (D1) from the Introduction. $\square$

Lemma 2.5. *Let* $\mathbf{A}$ *be an MV-algebra,* s, t *semi-states on* $\mathbf{A}$*. Then* $t \leq s$ *iff* $t(x) = 1$ *implies* $s(x) = 1$ *for all* $x \in A$*.*

Proof. Clearly, $t \leq s$ yields the condition $t(x) = 1$ implies $s(x) = 1$ for all $x \in A$.

Assume now that $t(x) = 1$ implies $s(x) = 1$ for all $x \in A$ is valid and that there is $y \in A$ such that $s(y) < t(y)$. Thus, there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $s(y) < r < t(y)$. By Corollary 2.1 there is a term t_r in $T_{\mathbb{D}}$ such that $t_r(s(y)) < 1$ and $t_r(t(y)) = 1$. It follows that $s(t_r(y)) = t_r(s(y)) < 1$ and $t(t_r(y)) = t_r(t(y)) = 1$. The last condition yields that $s(t_r(y)) = 1$, a contradiction. □

Proposition 2.3. *Let* $\mathbf{A}$ *be an MV-algebra,* t *a semi-state on* $\mathbf{A}$ *and* $S_t = \{s : \mathbf{A} \to [0,1] \mid \ s \text{ is an MV-morphism}, s \geq t\}$*. Then* $t = \bigwedge S_t$*.*

Proof. Clearly, $t \leq \bigwedge S_t$. Assume that there is $x \in A$ such that $t(x) < \bigwedge S_t(x)$. Thus, there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t(x) < r < \bigwedge S_t(x)$. Again by Corollary 2.1 there is a term t_r in $T_{\mathbb{D}}$ such that $t(t_r(x)) = t_r(t(x)) < 1$. Let us put $F = \{z \in A \mid t(z) = 1\}$. The set F is by the condition (iii) a filter of $\mathbf{A}$, $t_r(x) \notin F$. Hence there is by Proposition 2.2 an MV-morphism $s : \mathbf{A} \to [0,1]$ such that $s(F) \subseteq \{1\}$ and $s(x) < r < 1$. It follows by Lemma 2.5 that $t \leq s$, i.e., $s \in S_t$ and $s(x) < r < \bigwedge S_t(x) \leq s(x)$, a contradiction. □

Corollary 2.3. *Any semi-state on an MV-algebra* $\mathbf{A}$ *is a strong semi-state.*

Corollary 2.4. *The only semi-state* s *on an MV-algebra* $\mathbf{A}$ *with* $s(0) \neq 0$ *is the constant function* $s(x) = 1$ *for all* $x \in A$*.*

Corollary 2.5. *The only semi-state* s *on the standard MV-algebra* $[0,1]$ *with* $s(0) = 0$ *is the identity function.*

Remark 2.8.

1. It is transparent that all the preceding notions and results including Proposition 2.2 can be dualized. In particular, any dual semi-state, i.e., a map $s : \mathbf{A} \to [0,1]$ satisfying conditions (i),(ii),(iv), (v) and the

dual condition (iii)' $s(x) = 0$ and $s(y) = 0$ implies $s(x \oplus y) = 0$ is a join of extremal states on $\mathbf{A}$.

2. Let us denote by $\mathcal{S}_{\mathbf{A}}$ the set of all semi-states on an MV-algebra $\mathbf{A}$. Then $\mathcal{S}_{\mathbf{A}}$ is a $\bigwedge$-complete sub-semi-lattice of the complete lattice $[0,1]^A$ such that extremal states (morphism of MV-algebras into $\mathbf{A}$) are maximal elements of $\mathcal{S}_{\mathbf{A}}$. Similarly, the set $\mathcal{DS}_{\mathbf{A}}$ of all dual semi-states on an MV-algebra $\mathbf{A}$ is a $\bigvee$-complete sub-semi-lattice of the complete lattice $[0,1]^A$ such that extremal states are minimal elements of $\mathcal{DS}_{\mathbf{A}}$.

3. Proposition 2.3 immediately yields that two points of an MV-algebra $\mathbf{A}$ are separated by extremal states if and only if they are separated by semi-states if and only if they are separated by states.

Proposition 2.4. *Let* $\mathbf{A}$ *be an MV-algebra,* s *a state on* $\mathbf{A}$*. Then the following conditions are equivalent:*

(a) s *is a morphism of MV-algebras,*

(b) s *satisfies the condition* $s(x \wedge \neg x) = s(x) \wedge s(\neg x)$ *for all* $x \in A$*,*

(c) s *satisfies the condition (v) from Definition 2.1,*

(d) s *satisfies the condition (iv) from Definition 2.1,*

(e) s *satisfies the condition (viii) from Definition 2.1.*

Proof. (a) $\Rightarrow$ (b): It is evident.

(b) $\Rightarrow$ (c): Let $x \in A$. Then by (D3) $s(x \oplus x) = s(x \oplus (x \wedge \neg x))$ and $x \odot (x \wedge \neg x) = 0$. Since s is a state and (b) is valid we get that $s(x \oplus (x \wedge \neg x)) = s(x) + s(x \wedge \neg x) = s(x) + s(x) \wedge s(\neg x) = s(x) + s(x) \wedge \neg s(x) = s(x) \oplus s(x)$.

(c) $\Rightarrow$ (d): For any state the condition (v) is equivalent with (iv).

(d) $\Rightarrow$ (e): Clearly, any state satisfies conditions (i) and (ii) from Definition 2.1. Let us check the condition (iii). Assume that $s(x) = 1 = s(y)$.

Then $s(\neg x) = 0 = s(\neg y)$ and hence again by (D3) $s(x \odot y) = \neg s(\neg x \oplus \neg y) = \neg s(\neg x \oplus (x \wedge \neg y)) = \neg(s(\neg x) + s(x \wedge \neg y)) = \neg(0 + 0) = 1$.

By the assumption (b) we have that the condition (iv) is satisfied and for any state the condition (v) is equivalent with (iv). It follows that s is a semi-state. By Corollary 2.3 s is a strong semi-state, i.e., (viii) is satisfied.

(e) $\Rightarrow$ (a): It follows from [131, Lemma 3.1]. □

Remark 2.9. In particular, it follows from Proposition 2.4 that the only states which are also semi-states are extremal ones.

Definition 2.2. Let P, Q be bounded posets and let S be a set of order-preserving maps from P to Q. Then

(i) S is called *order determining* if

$$((\forall s \in S)\, s(a) \leq s(b)) \implies a \leq b$$

for any elements $a, b \in P$;

(ii) S is called *strongly order determining* if

$$((\forall s \in S)\, s(a) = 1 \implies s(b) = 1) \implies a \leq b$$

for any elements $a, b \in P$.

Note that Greechie in [94] proved that there is a finite orthomodular lattice that has an order determining set of states but there is no strongly order determining set of states on it. We will show that in the setting of MV-algebras the situation is completely different.

Proposition 2.5. *Let* $\mathbf{A}$ *be an MV-algebra,* S *a set of semi-states on* $\mathbf{A}$*,* $S^{\mathbb{D}} = \{s \circ t_r \mid s \in S, r \in (0,1) \cap \mathbb{D}\}$*. Then the following conditions are equivalent:*

(a) $S^{\mathbb{D}}$ *is strongly order determining,*

(b) $((\forall s \in S, r \in (0,1) \cap \mathbb{D})\, s(a) \geq r \implies s(b) \geq r) \implies a \leq b$ *for any elements* $a, b \in A$*.*

(c) S is order determining.

Proof. (a) $\iff$ (b): A fortiori by Corollary 2.1.
(b) $\Rightarrow$ (c): Assume that (b) holds. Choose $x, y \in A$ such that, for all $s \in S$, $s(x) \leq s(y)$. Let $r \in (0,1) \cap \mathbb{D}, s(x) \geq r$. Then $s(y) \geq r$. This yields by (b) that $x \leq y$.
(c) $\Rightarrow$ (a): Assume that S is order determining and that there are $x, y \in A$, $x \not\leq y$ such that, for all $s \in S$, $r \in (0,1) \cap \mathbb{D}$ $s(t_r(x)) = 1 \implies s(t_r(y)) = 1$. It follows that there is $t \in S$ such that $t(x) > t(y)$. Thus, there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t(y) < r < t(x)$. This yields that, by Corollary 2.1, there is a term t_r in $T_{\mathbb{D}}$ such that $t_r(t(y)) < 1$ and $t(t_r(x)) = t_r(t(x)) = 1$. Therefore $1 = t(t_r(y)) = t_r(t(y)) < 1$, a contradiction. $\square$

Corollary 2.6. *Let* **A** *be an MV-algebra, S a set of MV-morphism from* **A** *to* $[0,1]$. *Then the following conditions are equivalent:*

(a) $S^{\mathbb{D}}$ is strongly order determining,

(b) S is order determining.

Note that

1. In the case of orthomodular lattices we have, for any set S of order-preserving maps, that $S = S^{\mathbb{D}}$.

2. In the case of MV-algebras, an MV-algebra possesses an order determining set of semi-states iff it possesses an order determining set of extremal states.

2.2.4 Functions between MV-algebras and their construction

This section studies the notion of an fm-function between MV-algebras (strong fm-function between MV-algebras). The main purpose of this section is to establish in some sense a canonical construction of strong fm-function between MV-algebras. This construction is an ultimate source of numerous examples.

Definition 2.3. By an *fm-function between MV-algebras* G is meant a function $G : \mathbf{A}_1 \to \mathbf{A}_2$ such that $\mathbf{A}_1 = (A_1; \oplus_1, \neg_1, 0_1)$ and $\mathbf{A}_2 = (A_2; \oplus_2, \neg_2, 0_2)$ are MV-algebras and

(FM1) $G(1_1) = 1_2$,

(FM2) $x \leq_1 y$ implies $G(x) \leq_2 G(y)$,

(FM3) $G(x) = 1_2 = G(y)$ implies $G(x \odot_1 y) = 1_2$,

(FM4) $G(x) \odot_2 G(x) = G(x \odot_1 x)$,

(FM5) $G(x) \oplus_2 G(x) = G(x \oplus_1 x)$.

If moreover G satisfies conditions

(FM6) $G(x) \odot_2 G(y) \leq G(x \odot_1 y)$,

(FM7) $G(x) \oplus_2 G(y) \leq G(x \oplus_1 y)$,

(FM8) $G(x) \wedge_2 G(y) = G(x \wedge_1 y)$,

(FM9) $G(x^n) = G(x)^n$ for all $n \in \mathbb{N}$,

(FM10) $n \times_2 G(x) = G(n \times_1 x)$ for all $n \in \mathbb{N}$,

we say that G is a *strong fm-function between MV-algebras.*

If $G : \mathbf{A}_1 \to \mathbf{A}_2$ and $H : \mathbf{B}_1 \to \mathbf{B}_2$ are fm-functions between MV-algebras, then a *morphism between G and H* is a pair (φ, ψ) of morphisms of MV-algebras $\varphi : \mathbf{A}_1 \to \mathbf{B}_1$ and $\psi : \mathbf{A}_2 \to \mathbf{B}_2$ such that $\psi(G(x)) = H(\varphi(x))$, for any $x \in A_1$.

Note that (FM8) yields (FM2), (FM9) yields (FM4) and (FM10) yields (FM5). Also, a composition of fm-functions (strong fm-functions) is an fm-function (a strong fm-function) again and any morphism of MV-algebras is an fm-function (a strong fm-function).

The notion of an fm-function generalizes both the notions of a semi-state and of a (*strong*) $\odot$-*operator* from [127] which is a (strong) fm-function G from $\mathbf{A}_1$ to itself such that (FM6) is satisfied.

We say that an $\odot$-operator G is *contractive* (*extensive, transitive*) if $G(x) \leq x$ ($x \leq G(x)$, $G(x) \leq G(G(x))$) for all $x \in A_1$. An $\odot$-operator G that is both contractive and transitive is called a *conucleus.*

Example 2.2. Here are some simple examples of MV-algebras with an $\odot$-operator.

(1) The identity map on any MV-algebra $\mathbf{A}$ is a strong operator that is contractive, extensive and transitive.

(2) The constant map $\mathbf{1} : \mathbf{A} \to \mathbf{A}$ defined by $\mathbf{1}(x) = 1$ for all $x \in A$ is a strong operator that is both extensive and transitive but it is not contractive.

(3) The map $\Box : [0,1]^3 \to [0,1]^3$ given by $(x, y, z) \mapsto (y \wedge z, x \wedge z, x \wedge y)$ is a strong operator on the MV-algebra $[0,1]^3$ that is neither contractive nor extensive nor transitive.

(4) If $\mathcal{C}$ denotes Chang's MV-algebra and if k is a positive integer then the map $\Box_k : \mathcal{C} \to \mathcal{C}$ given by $\Box_k(x) = k \times x$ is a strong operator on $\mathcal{C}$ that is both extensive and transitive but it is not contractive.

According to both (FM4) and (FM5), $G|_{B(\mathbf{A}_1)} : B(\mathbf{A}_1) \to B(\mathbf{A}_2)$ is an fm-function (a strong fm-function) whenever G has the respective property.

Lemma 2.6. *Let $G : \mathbf{A}_1 \to \mathbf{A}_2$ be an fm-function between MV-algebras, $r \in (0,1) \cap \mathbb{D}$. Then $t_r(G(x)) = G(t_r(x))$ for all $x \in A_1$.*

Proof. Note that $G(x) \oplus_2 G(x) = G(x \oplus_1 x)$ by (FM5) and $G(x) \odot_2 G(x) = G(x \odot_1 x)$ by (FM4). Then, since $t_r \in T_{\mathbb{D}}$ is defined inductively using only the operations $(-) \oplus (-)$ and $(-) \odot (-)$, we get $t_r(G(x)) = G(t_r(x))$. $\Box$

By a *frame* is meant a triple (S, T, R) where S, T are non-void sets and $R \subseteq S \times T$. If $S = T$ we say that the pair (T, R) is a *time frame.* Having an MV-algebra $\mathbf{M} = (M; \oplus, \odot, \neg, 0, 1)$ and a non-void set T, we can produce the direct power $\mathbf{M}^T = (M^T; \oplus, \odot, \neg, o, j)$ where the operations $\oplus$, $\odot$ and $\neg$ are defined and evaluated on $p, q \in M^T$ componentwise. Moreover, o, j

are such elements of M^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathbf{M}^T$ is again an MV-algebra.

The notion of frame allows us to construct new examples of MV-algebras with a strong operator.

Theorem 2.16. *Let* $\mathbf{M}$ *be a linearly ordered complete MV-algebra,* (S, T, R) *be a frame and* G^* *be a map from* M^T *into* M^S *defined by*

$$G^*(p)(s) = \bigwedge\{p(t) \mid t \in T, sRt\},$$

for all $p \in M^T$ *and* $s \in S$*. Then* G^* *is a strong fm-function between MV-algebras which has a left adjoint* P^**. In this case, for all* $q \in M^S$ *and* $t \in T$*,*

$$P^*(q)(t) = \bigvee\{q(s) \mid s \in T, sRt\}$$

and $P^* : (\mathbf{M}^S)^{op} \to (\mathbf{M}^T)^{op}$ *is a strong fm-function between MV-algebras.*

Proof. Trivially we can verify $G^*(j_{\mathbf{M}^T}) = j_{\mathbf{M}^S}$ due to the fact that $j_{\mathbf{M}^T}(t) = 1$ for each $t \in T$ thus (FM1) holds.

Moreover, for any $p \in M^T$ and $q \in M^S$, we can compute:

$$\begin{aligned} q(s) \le G^*(p)(s) \text{ for all } s \in T &\Longleftrightarrow q(s) \le \bigwedge\{p(t) | t \in T, sRt\} \text{ for all } s \in T \\ &\Longleftrightarrow q(s) \le p(t) \text{ for all } s, t \in T, sRt \\ &\Longleftrightarrow \textstyle\bigvee_{i \in I}\{q(s) | s \in T, sRt\} \le p(t) \text{ for all } t \in T \\ &\Longleftrightarrow P^*(q)(t) \le p(t) \text{ for all } t \in T. \end{aligned}$$

This yields that $q \le G^*(p)$ iff $P^*(q) \le p$. Then P^* is a left adjoint of G^*. Hence G^* preserves arbitrary meets and hence the conditions (FM2) and (FM8) are satisfied.

The conditions (FM3)-(FM7) and (FM9)-(FM10) can be easily shown by the same considerations as in Lemma 2.4. □

We say that $G^* : \mathbf{M}^T \to \mathbf{M}^S$ is the *canonical strong fm-function between MV-algebras induced by the frame* (S, T, R) *and the MV-algebra* $\mathbf{M}$.

Theorem 2.17. *Let* $\mathbf{M}$ *be a linearly ordered complete MV-algebra,* (T, R) *be a time frame and* G^* *be a map from* M^T *into itself defined by*

$$G^*(p)(s) = \bigwedge\{p(t) \mid t \in T, sRt\},$$

for all $p \in M^T$ *and* $s \in T$*. Then* $(\mathbf{M}^T; G^*)$ *is an MV-algebra with a strong operator* G^* *which has a left adjoint* P^**. In this case, for all* $p \in M^T$ *and* $t \in T$*,*

$$P^*(p)(t) = \bigvee\{p(s) \mid s \in T, sRt\}$$

and $((\mathbf{M}^{op})^T; P^*)$ *is an MV-algebra with a strong operator* P^**. Moreover, the following holds:*

(a) *If* R *is reflexive then* G^* *is contractive.*

(b) *If* R *is transitive then* G^* *is transitive.*

(c) *If* R *is both reflexive and transitive then* G^* *is a conucleus.*

Proof. The first part follows from Theorem 2.16. Let us check (a). Since R is reflexive then from tRt we obtain that $G^*(p)(t) = \bigwedge\{p(t) \mid t \in T, sRt\} \leq p(t)$ for any $p \in M^T$. Let us proceed similarly for (b). We have, for all $s \in T$,

$$\begin{aligned} G^*(p)(s) &= \bigwedge\{p(u) \mid u \in T, sRu\} \leq \bigwedge\{p(u) \mid t, u \in T, sRt, tRu\} \\ &= \bigwedge\{\bigwedge\{p(u) \mid u \in T, tRu\} \mid t \in T, sRt\} \\ &= \bigwedge\{G^*(p)(t) \mid t \in T, sRt\} = G^*(G^*(p))(s) \end{aligned}$$

since $\{u \in T \mid t \in T, sRt, tRu\} \subseteq \{u \in T \mid sRu\}$ by transitivity. The validity of (c) follows immediately from (a) and (b).

$\square$

From the preceding immediately follows Theorem 2.15 that any time frame induces its own couple of tense operators. As the next step we will prove that any couple of tense operators on any semisimple MV-algebra can be embedded into $([0, 1]^T, G^*, H^*)$ where G^* and H^* are a tense operators induced by some time frame (T, ρ).

2.2.5 The main theorem and its applications

Before proving our main theorem, we remark that *semisimple* MV-algebras [46] are just subdirect products of the simple MV-algebras. Any simple MV-algebra is uniquely embeddable into the standard MV-algebra on the interval $[0,1]$ of reals. It is known that an MV-algebra is semisimple if and only if the intersection of the set of its maximal (prime) filters is equal to the set $\{1\}$. Note also that any complete MV-algebra is semisimple.

Hence a semisimple MV-algebra $\mathbf{A}$ can be embedded into $[0,1]^T$ (see [4]) where T is the set of all ultrafilters (morphisms into the standard MV-algebra) and $\pi_F(x) = x(F) = x/F \in [0,1]$ for any $x \in \mathbf{A} \subseteq [0,1]^T$ and any $F \in T$; here $\pi_F : [0,1]^T \to [0,1]$ is the respective projection onto $[0,1]$.

Theorem 2.18. *Let $G : \mathbf{A}_1 \to \mathbf{A}_2$ be an fm-function between semisimple MV-algebras, T a set of all MV-morphism from $\mathbf{A}_1$ to the standard MV-algebra $[0,1]$ and S a set of all MV-morphism from $\mathbf{A}_2$ to $[0,1]$.*

Further, let (S, T, ρ_G) be a frame such that the relation $\rho_G \subseteq S \times T$ is defined by

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in A_1.$$

Then the fm- function G is representable via the canonical strong fm-function $G^*\colon [0,1]^T \to [0,1]^S$ between MV-algebras induced by the frame (S, T, ρ_G) and the standard MV-algebra $[0,1]$, i.e., the following diagram of fm-functions commutes:

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r, "G"] \arrow[d, "i^T_{\mathbf{A}_1}"'] & \mathbf{A}_2 \arrow[d, "i^S_{\mathbf{A}_2}"] \\
{[0,1]^T} \arrow[r, "G^*"'] & {[0,1]^S}
\end{tikzcd} .$$

Proof. Assume that $x \in A_1$ and $s \in S$. Then $i^S_{\mathbf{A}_2}(G(x))(s) = s(G(x)) \leq t(x)$ for all $t \in T$, $(s,t) \in \rho_G$. It follows that $i^S_{\mathbf{A}_2}(G(x)) \leq G^*(i^T_{\mathbf{A}_1}(x))$.

Note that $s \circ G$ is a semi-state on $\mathbf{A}_1$ and by Proposition 2.3 we get that

$$\begin{aligned} s \circ G &= \bigwedge\{t : \mathbf{A}_1 \to [0,1] \mid \ t \text{ is an MV-morphism}, t \geq s \circ G\} \\ &= \bigwedge\{t \in T \mid \ (s,t) \in \rho_G\}. \end{aligned}$$

This yields that actually $i^S_{\mathbf{A}_2}(G(x)) = G^*(i^T_{\mathbf{A}_1}(x))$. □

Proposition 2.6. *For any MV-algebra $\mathbf{A}_1$, any semisimple MV-algebra $\mathbf{A}_2$ with a set S of all MV-morphism from $\mathbf{A}_2$ to $[0,1]$ and any map $G : A_1 \to A_2$ the following conditions are equivalent:*

(i) *G is an fm-function between MV-algebras.*

(ii) *G is a strong fm-function between MV-algebras.*

Proof. (i) $\Longrightarrow$ (ii): Note that the composition $\pi_s \circ i^S_{\mathbf{A}_2} \circ G$ is a strong semi-state for any $s \in T$. It follows that $i^S_{\mathbf{A}_2} \circ G$ is a strong fm-function between MV-algebras. Since the embedding $i^S_{\mathbf{A}_2} : \mathbf{A}_2 \to [0,1]^S$ reflects order we obtain that conditions (FM6)-(FM10) are satisfied.
(ii) $\Longrightarrow$ (i): It is evident. □

The above proposition leads to a natural question whether there is a difference between fm-functions and strong fm-functions. The following proposition shows that the classes of fm-functions and strong fm-functions are different.

Proposition 2.7. *There is an MV-algebra $\mathbf{A}$ with an fm-function G on $\mathbf{A}$ such that G is not a strong fm-function and G is not an $\odot$-operator on $\mathbf{A}$.*

Proof. Let $\mathbf{A}$ be the Chang MV-algebra $\mathbf{C} = \{0, c, c\oplus c, c\oplus c\oplus c, \dots, \dots, 1\ominus (c\oplus c\oplus c), 1\ominus(c\oplus c), 1\ominus c, 1\}$ (for more details see e.g. [73, p. 141]. Note only that we define $0 \times c = 0$, $n \times c = c \oplus \cdots \oplus c$, $1 \ominus 0 \times c = 1$ and $1 \ominus n \times c = ((\dots(1\ominus c)\ominus \dots)\ominus c)$. Also

$$x \odot y = \begin{cases} 0 & \text{if } x = n\times c, y = m \times c \\ \max(0,(n-m)) \times c & \text{if } x = n \times c, y = 1\ominus m\times c \\ \max(0,(m-n))\times c & \text{if } x = 1\ominus n\times c, y = m\times c \\ 1\ominus(m+n)\times c & \text{if } x = 1\ominus n\times c, y = 1\ominus m\times c \end{cases}$$

and

$$x \oplus y = \begin{cases} (m+n) \times c & \text{if } x = n \times c, y = m \times c \\ 1 \ominus \max(0, (m-n)) \times c & \text{if } x = n \times c, y = 1 \ominus m \times c \\ 1 \ominus \max(0, (n-m)) \times c & \text{if } x = 1 \ominus n \times c, y = m \times c \\ 1 & \text{if } x = 1 \ominus n \times c, y = 1 \ominus m \times c. \end{cases}$$

It follows that $(n \times c) \odot (n \times c) = 1$ and $(1 \ominus n \times c) \oplus (1 \ominus n \times c) = 1$ for all nonnegative integers n.

Let r be any positive integer. Any positive integer n can be uniquely written as follows: $n = 2^k m$ such that $2 \not| m$, $k \geq 0$ is a suitable integer.

Let us define a function $G_r : \mathbf{C} \to \mathbf{C}$ by the following prescription

$$G_r(0) = 0, G_r(c) = r \times c,$$

$$G_r(n \times c) = \begin{cases} 2^k G_r(m \times c) & \text{if } n = 2^k m \\ G_r((n-1) \times c) \oplus c & \text{otherwise,} \end{cases}$$

$$G_r(1 \ominus n \times c) = 1 \ominus G_r(n \times c).$$

It is easy to check that G_r is an fm-function for all positive integers r and, e.g. for $r = 2$, G_2 does not satisfy (FM6) and (FM10). Namely, we have

$$G_2(0) = 0, G_2(c) = 2 \times c, G_2(2 \times c) = 4 \times c, G(3 \times c) = 5 \times c,$$
$$G(4 \times c) = 8 \times c, G(5 \times c) = 9 \times c, \dots.$$

It follows that $G_2(4 \times c) \odot G_2(1 \ominus 3 \times c) = 8.c \odot (1 \ominus 5 \times c) = 3 \times c \not\leq G_2(4 \times c \odot (1 \ominus 3 \times c)) = G_2(c) = 2 \times c$ and $G_2(3 \times c) = 5 \times c \neq 6 \times c)$. Therefore G_2 is not a strong fm-function and G_2 is not an $\odot$-operator on $\mathbf{C}$.

Now, let us check that G_r is an fm-function for all positive integers r.

(FM1) $G_r(1) = 1 \ominus G_r(0) = 1 \ominus 0 = 1$.

(FM2) Let $x, y \in \mathbf{C}$. It is enough to check that $x < y$ yields $G_r(x) < G_r(y)$.

(a) Let $y = n_2 \times c$, $n_2 = 2^{k_2} m_2$ be a nonnegative integer such that $2 \not| m_2$, $k_2 \geq 0$ is a suitable integer.
We will proceed by induction with respect to n_2. Let $x < y$. Then $x = n_1 \times c$, $n_1 = 2^{k_1} m_1$ such that $2 \not| m_1$, $k_1 \geq 0$ is a suitable integer.
If $n_2 = 1$ then $n_1 = 0$, i.e., $x = 0$ and $y = c$. It follows that $G_r(x) = 0 < r \times c = G_r(y)$.
Now, assume that $n_2 > 1$ and the statement holds for all $n_3 < n_2$, i.e., $n_1 \times c < n_3 \times c < n_2 \times c$ implies $G_r(n_1 \times c) < G_r(n_3 \times c)$.
We put $d = |k_1 - k_2|$ and we will proceed by induction with respect to d. If $d = 0$ then $k_1 = k_2$. If $x = 0$ we are finished. Let $x > 0$. Then also $0 < m_1 < m_2$ and $G_r((m_1 - 1) \times c) < G_r((m_2 - 1) \times c)$ since $(m_2 - 1) \times c < n_2$. This yields $G_r(x) = 2^{k_1} \times G_r(m_1 \times c) = 2^{k_1} \times (G_r((m_1 - 1) \times c) \oplus c) < 2^{k_1} \times (G_r((m_2 - 1) \times c) \oplus c) = 2^{k_2} \times G_r(m_2 \times c) = G_r(y)$.
Assume now that $d > 1$. If $x = 0$ we are finished. Let $x > 0$. If $k_2 = k_1 + d$ then we obtain that $0 < m_1 < 2^d m_2$. Then also $2m_3 = m_1 - 1 < 2^d m_2$, i.e., $m_3 < 2^{d-1} m_2$. By the induction hypothesis we get that $G_r(m_3 \times c) < 2^{d-1} \times G_r(m_2 \times c)$. It follows that $2G_r(m_3 \times c) = G_r((2m_3) \times c) < G_r((2m_3) \times c) \oplus c = G_r((2m_1) \times c) < 2^d \times G_r(m_2 \times c)$.
If $k_1 = k_2 + d$ then we have that $0 < 2^d m_1 < m_2$. By the induction hypothesis with respect to n_2 we have that $G_r(2^d \times m_1) \leq G_r((m_2 - 1) \times c) < G_r(m_2 \times c)$

(b) Let $y = 1 \ominus n_2 \times c$, n_2 be a nonnegative integer and let $x < y$, $x \in \mathbf{C}$.
If $x = n_1 \times c$ then $G_r(x) = m_1 \times c$ and $G_r(y) = 1 \ominus m_2 \times c$ for suitable nonnegative integers m_1, m_2. It follows that $G_r(x) < G_r(y)$.
If $x = 1 \ominus n_1 \times c$ then $n_2 < n_1$. From (a) we get that $G_r(n_2 \times c) < G_r(n_1 \times c)$. Therefore $G_r(x) = 1 \ominus G_r(n_1 \times c) < 1 \ominus G_r(n_2 \times c) = G_r(y)$ and we are done.

(FM3) Let $x, y \in \mathbf{C}$, $G_r(x) = 1 = G_r(y)$. Then, from the definition of G_r, we get that $x = 1 = y$. It follows that $x \odot y = 1$, i.e., $G_r(x \odot y) = 1$.

(FM4) Let $x \in \mathbf{C}$. If $x = 1 \ominus n \times c$ for some nonnegative integer n then $G_r(x) = 1 \ominus m \times c$ for some nonnegative integer m. It follows that $G_r(x) \odot G_r(x) = 1 \ominus (2m) \times c$ and $G_r(x \odot x) = G_r(1 \ominus (2n) \times c)$. But $G_r((2n) \times c) = 2 \times G_r(n \times c) = 2(m \times c)$, i.e., $G_r(x \odot x) = G_r(x) \odot G_r(x)$.

If $x = n \times c$ for some nonnegative integer n then $G_r(x) = m \times c$ for some nonnegative integer m. This yields that $G_r(x) \odot G_r(x) = 0$ and that $G_r(x \odot x) = G_r(0) = 0$.

(FM5) Let $x \in \mathbf{C}$. If $x = n \times c$ for some nonnegative integer n then $G_r(x) = m \times c$ for some nonnegative integer m. We have $G_r(x) \oplus G_r(x) = (2m) \times c$ and $G_r(x \oplus x) = G_r((2n) \times c) = 2 \times G_r(n \times c) = (2m) \times c$.

Now, let $x = 1 \ominus n.c$ for some nonnegative integer n. Then $G_r(x) = 1 \ominus m \times c$ for some nonnegative integer m. We obtain $G_r(x) \oplus G_r(x) = 1$ and $G_r(x \oplus x) = G_r(1) = 1$.

$\square$

Note that our approach of using semi-states in the above proof of Theorem 2.18 also covers the main result of the paper [127] which is Theorem 4.5 from [127].

Theorem 2.19. [127, Theorem 4.5] **(Representation theorem for MV-algebras with an $\odot$-operator)** *For any semisimple MV-algebra* $\mathbf{M}$ *with an $\odot$-operator G,* $\mathbf{M}$ *is embeddable via MV-operator morphism $i^T_{\mathbf{M}}$ into the canonical MV-algebra $\mathcal{L}_G = ([0,1]^T; G^*)$ with a strong fm-operator G^* induced by the canonical frame (T, ρ_G) and the standard MV-algebra* $[0,1]$*. Further, for all $x \in M$ and for all $s \in T$, $s(G(x)) = G^*((t(x))_{t \in T})(s)$.*

The next theorem, which is a solution of a half of the Open problem 5.1 in [54], was first proved by the first author of the paper for tense operators satisfying (FM9) and (FM10) and then, based on the idea of the first author, proved in the full generality by the second author as Theorem 5.5 from [127], can be written as follows.

Theorem 2.20. *Let* $\mathbf{M}$ *be a semisimple MV-algebra with tense operators* G *and* H*. Then* $(\mathbf{M}, G, H)$ *can be embedded into the tense MV-algebra* $([0,1]^T, G^*, H^*)$ *induced by the frame* (T, ρ_G)*, where* T *is the set of all maximal proper filters and the relation* ρ_G *is defined by*

$$A\rho_G B \text{ if and only if } G(x)/A \leq x/B \text{ for any } x \in M.$$

Proof. First, let us define a second relation $\rho_H \subseteq T^2$ by the stipulation:

$$B\rho_H A \text{ if and only if } H(x)/B \leq x/A \text{ for any } x \in M.$$

Claim 1. *The equality* $\rho_G = \rho_H^{-1}$ *holds.*

Proof. Let us suppose that $A\rho_G B$ for some $A, B \in T$. Due to Definition 2.1 vi) we have $\neg G\neg H(x) \leq x$ and $\neg x/A \leq G\neg H(x)/A$. $A\rho_G B$ yields $G\neg H(x)/A \leq \neg H(x)/B$ and together $\neg x/A \leq \neg H(x)/B$ yields $H(x)/B \leq x/A$ for any $x \in M$.

Due to the definition of ρ_H we have $B\rho_H A$ and $\rho_G \subseteq \rho_H^{-1}$. Analogously we can prove the second inclusion.

□

The remaining part follows from Theorem 2.18. Basically, the obtained equations $G^*(x) = G(x)$ and $H^*(x) = H(x)$ finish the proof. □

Theorem 2.21. *a) If* $([0,1]^T, G^*, H^*)$ *is a tense MV-algebra induced by a time frame* (T, ρ)*, then*

(i) if ρ *is reflexive then* $G^*(x) \leq x$ *and* $H^*(x) \leq x$ *hold for any* $x \in [0,1]^T$*,*

(ii) if ρ *is symmetric then* $G^*(x) = H^*(x)$ *holds for any* $x \in [0,1]^T$*,*

(iii) if ρ *is transitive then* $G^*G^*(x) \geq G^*(x)$ *and* $H^*H^*(x) \geq H^*(x)$ *hold for any* $x \in [0,1]^T$*.*

b) Let $(\mathbf{M}, G, H)$ be a semisimple tense MV-algebra and (T, ρ_G) the time frame which induces the tense MV-algebra $([0,1]^T, G^*, H^*)$ by Theorem 2.20. Then

(i) if $G(x) \le x$ and $H(x) \le x$ hold for any $x \in M$ then ρ_G is reflexive,

(ii) if $G(x) = H(x)$ holds for any $x \in M$ then ρ_G is symmetric,

(iii) if $GG(x) \ge G(x)$ and $HH(x) \ge H(x)$ hold for any $x \in M$ then ρ_G is transitive.

Proof. ai) If the relation ρ is reflexive, then $i\rho i$ yields $G^*(x)(i) = \bigwedge_{i\rho j} x(j) \le x(i)$ for any $i \in T$. The part for H^* we can prove analogously.

aii) If ρ is symmetric then $G^*(x)(i) = \bigwedge_{i\rho j} x(j) = \bigwedge_{j\rho i} x(j) = H^*(x)(i)$ for any $i \in T$ which clearly yields $G^* = H^*$.

aiii) If ρ is transitive then $\{x(k) \,|\, i\rho j \text{ and } j\rho k\} \subseteq \{x(k) \,|\, i\rho k\}$ and then

$$\begin{aligned} G^*G^*(x)(i) &= \bigwedge_{i\rho j} G^*(x)(j) = \bigwedge_{i\rho j} \bigwedge_{j\rho k} x(k) \\ &= \bigwedge \{x(k) \,|\, i\rho j \text{ and } j\rho k\} \ge \bigwedge_{i\rho k} x(k) = G^*x(i) \end{aligned}$$

holds for any $i \in T$.

b) We remark that relation ρ_G in Theorem 2.20 is defined by

$$A\rho_G B \text{ if and only if } G(x)/A \le x/B \text{ for any } x \in M.$$

bi) If $G(x) \le x$ for any $x \in M$ then $G(x)/A \le x/A$ holds for any $x \in M$ and thus $A\rho_G A$. Together ρ_G is reflexive.

bii) The Claim 1 in the proof of Theorem 2.20 shows that $G(x)/A \le x/B$ for any $x \in M$ if and only if $H(x)/B \le x/A$ for any $x \in M$. If $G = H$ holds then $G(x)/A \le x/B$ for any $x \in M$ if and only if $G(x)/B \le x/A$ for any $x \in M$ and consequently $A\rho_G B$ holds if and only if $B\rho_G A$ holds. Thus the relation ρ_G is symmetric.

biii) Let us suppose that $G(x) \le GG(x)$ for any $x \in M$. If $A\rho_G B$ and $B\rho_G C$ hold then any $x \in M$ satisfies $G(x)/A \le GG(x)/A \le G(x)/B \le x/C$ which yields $A\rho_G C$. Thus the relation ρ_G is transitive. □

Remark 2.10. Note that one can extend the number of fm-functions between MV-algebras arbitrarily and our results remain valid. Similarly as for semi-states in Remark 2.8 we could introduce the notion of a dual (strong) fm-function and *all the preceding results would also remain valid in this dual setting.*

Remark 2.11. We have settled a half of the Open problem 5.1 in [54] using a more general approach of fm-functions. The remaining part asks about the existence of a representation theorem for any tense MV-algebra via Di Nola representation theorem for MV-algebras. Note that in general an ultraproduct of a complete linearly ordered MV-algebra need not be a complete lattice hence the formulation of the remaining part [54] has to be changed accordingly to this fact. We hope that our results will be a next step in obtaining a general representation theorem for tense MV-algebras.

We expect that our method can be easily applied to modal or similar operators that may be treated as universal quantifiers on various types of MV-algebras. Also we expect some applications towards the field of quantum logic.

Chapter 3

Lattice effect algebras

3.1 Introduction and preliminaries

Effect algebras belong among the so-called 'quantum structures', i.e. algebraic structures that are related to the logical foundations of quantum mechanics. Some well-established structures, such as MV-algebras and orthomodular lattices, can be regarded as subclasses of effect algebras. We dare not explain the physical background, and so we borrow the words of Foulis and Bennett, who introduced effect algebras in their paper [84]: 'The effects in a quantum-mechanical system form a partial algebra and a partially ordered set which is the prototypical example of the effect algebras ... ' In [109], independently of [84], Kôpka and Chovanec defined D-posets that are in fact equivalent to effect algebras, albeit the motivation behind D-posets was different. Both effect algebras and D-posets are partial algebras, but we proved in [36] that it is possible to make them into total algebras, and the class of algebras thus obtained is even equationally definable (it is a finitely based variety). However, this approach has a drawback: many non-isomorphic algebras in the variety determine the same effect algebra. The situation is much better in case of lattice-ordered effect algebras because if the underlying poset is a lattice, then there is a 'canonical totalization' giving a natural one-one correspondence between lattice effect algebras and the algebras derived from them. Thus we can identify the class of lattice-ordered effect algebras with a certain variety, say $\mathcal{E}$, which contains the variety of MV-algebras as well as the variety (term equivalent to the variety) of orthomodular lattices, and is contained in the variety of our 'basic algebras' that we defined in [35]. In what follows, we study subvarieties of the variety $\mathcal{DE}$ that is defined—relative to $\mathcal{E}$—by lattice distributivity. We axiomatize all finitely generated subvarieties of $\mathcal{DE}$ (Section 3.2.1) and then describe the free algebras in these varieties (Section 3.2.2).

Though we assume familiarity with the basics of effect algebras and MV-algebras, we briefly explain all necessary concepts that are used here; our standard reference is [73] for effect algebras, and [46] for MV-algebras.

First, the definition.

By an *effect algebra* is meant a structure $\mathbf{E} = (E; +, 0, 1)$ where 0 and 1 are distinguished elements of E, $0 \neq 1$, and $+$ is a partial binary operation on E satisfying the following axioms for $p, q, r \in E$:

(E1) if $p + q$ is defined then $q + p$ is defined and $p + q = q + p$

(E2) if $q + r$ is defined and $p + (q + r)$ is defined then $p + q$ and $(p + q) + r$ are defined and $p + (q + r) = (p + q) + r$

(E3) for each $p \in E$ there exists a unique $p' \in E$ such that $p + p' = 1$; p' is called a *supplement* of p

(E4) if $p + 1$ is defined then $p = 0$.

In every effect algebra $\mathbf{E}$ we can introduce the *induced order* $\leq$ on E and the partial operation $-$ as follows

$$x \leq y \quad \text{if for some } z \in E \quad x + z = y, z = y - x$$

(see e.g. [73] for details). Then $(E; \leq)$ is an ordered set and $0 \leq x \leq 1$ for each $x \in E$.

A *lattice effect algebra*, or a *lattice-ordered effect algebra*, is one which is a lattice with respect to its natural ordering. If the lattice is distributive, we speak of *distributive lattice effect algebras*.

It is worth noticing that $a+b$ exists in an effect algebra $\mathbf{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal). Dually, we have a partial operation $\cdot$ on E such that $a \cdot b$ exists in an effect algebra $\mathbf{E}$ if and only if $a' \leq b$ in which case $a \cdot b = (a' + b')'$. This allows us to equip E with a dual effect algebraic operation such that $\mathbf{E}^{op} = (E; \cdot, 1, 0)$ is again an effect algebra, ${}'^{\mathbf{E}^{op}} = {}'^{E} = {}'$ and $\leq_{\mathbf{E}^{op}} = \leq^{op}$.

A *morphism of effect algebras* is a map between them such that it preserves the partial operation $+$, the bottom and the top elements. A map $s : E \to [0,1]$ is called a *state* on $\mathbf{E}$ if $s(0) = 0$, $s(1) = 1$ and $s(x+y) = s(x) + s(y)$ whenever $x + y$ exists in $\mathbf{E}$. Note that the real unit interval $[0,1]$ is an effect algebra such that $x + y$ exists in $[0,1]$ if and only if the usual sum of x and y is less or equal to 1.

A *morphism* $f : P_1 \to P_2$ *of bounded posets* is an order, top element and bottom element preserving map. Any morphism of effect algebras is a morphism of corresponding bounded posets. A morphism $f : P_1 \to P_2$ of bounded posets is *order reflecting* if, for all $a, b \in P_1$,

$$f(a) \leq f(b) \text{ if and only if } a \leq b.$$

An *isomorphism of effect algebras* is a surjective order reflecting morphism of effect algebras. In particular, $' : \mathbf{E} \to \mathbf{E}^{op}$ is an isomorphism of effect algebras.

On any lattice effect algebra $\mathbf{E}$ we may introduce total operations $\oplus$ and $\odot$ as follows: $x \oplus y = (x \wedge y') + y$ and $x \odot y = (x' \oplus y')'$.

An *E-morphism between lattice effect algebras* is a morphism $f : \mathbf{E}_1 \to \mathbf{E}_2$ of effect algebras such that

(i) $f(x \oplus x) = f(x) \oplus f(x)$.

Note that from the condition (i) we automatically have the following condition:

(ii) $f(x \odot x) = f(x) \odot f(x)$.

Any isomorphism of lattice effect algebras is an E-morphism which preserves lattice operations as well.

Prominent examples of lattice effect algebras include orthomodular lattices and MV-algebras:

Orthomodular lattices can be identified with lattice effect algebras satisfying the condition that $x + x$ is defined only if $x = 0$, because when $(E; \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, then $(E; +, 0, 1)$ (here $+$ is the restriction of $\vee$ to the pairs x, y with $x \leq y'$) is a lattice effect algebra that satisfies the condition.

MV-algebras are equivalent to the so-called *MV-effect algebras*, i.e. lattice effect algebras where $(x \vee y)' + y = x' + (x \wedge y)$ for all x, y. Specifically, if $(E; \oplus, \neg, 0)$ is an MV-algebra, then $(E; +, 0, 1)$ (here $+$ is the restriction of $\oplus$ to the pairs x, y with $x \leq \neg y$) is an MV-effect algebra in which $x' = \neg x$, and the total addition $\oplus$ can be recovered from the partial addition $+$ by $x \oplus y = (x \wedge y') + y$. On the other hand, given an MV-effect algebra, letting

$$x \oplus y = (x \wedge y') + y \quad \text{and} \quad \neg x = x' \tag{3.1}$$

we obtain an MV-algebra in which $x + y = x \oplus y$ for $x \leq \neg y$. It follows that any linearly ordered effect algebra is an MV-algebra.

It is known that the variety of MV-algebras is generated by the *standard MV-algebra* $([0, 1]; \oplus, \neg, 0)$. In the corresponding MV-effect algebra $([0, 1]; +, 0, 1)$, the partial $+$ is just the restriction to $[0, 1]$ of the usual addition of reals. Finite subalgebras of $([0, 1]; \oplus, \neg, 0)$ are important, too. The *standard n-element MV-chain* is the subalgebra of $([0, 1]; \oplus, \neg, 0)$ with the universe $C_n = \big\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\big\}$.

Another crucial example is the so-called *(distributive) diamond*:

Let $D = \{0, a, b, 1\}$ be equipped with $+$ as follows:

$+$	0	a	b	1
0	0	a	b	1
a	a	1	.	.
b	b	.	1	.
1	1	.	.	.

Then $(D; +, 0, 1)$ is a distributive lattice effect algebra where $a' = a$ and $b' = b$; the induced lattice is distributive (a and b are atoms). We should note that $(D; +, 0, 1)$ is not an MV-effect algebra since $(a \vee b)' + b = b$ while $a' + (a \wedge b) = a$.

The significance of the diamond follows from the following result (see [95, Theorem 7.9]):

> *All finite distributive lattice effect algebras are direct products of finite chains and diamonds.*

Here, 'finite chains' are the MV-effect algebras $(C_n; +, 0, 1)$ corresponding to finite subalgebras of the standard MV-algebra, and direct products of effect algebras are defined in a natural manner, i.e. $(x_1, \ldots, x_s) + (y_1, \ldots, y_s)$ is defined and equals $(x_1 + y_1, \ldots, x_s + y_s)$ provided that $x_i + y_i$ is defined for every $i \in \{1, \ldots, s\}$. Therefore, since the focus is on distributive lattice effect algebras, it is clear that the effect algebras $(C_n; +, 0, 1)$ and $(D; +, 0, 1)$ will play a central role in our exposition.

Let us return to (3.1). Using this rule, all lattice effect algebras, not only MV-effect algebras, can be made total algebras; that is, with an arbitrary lattice effect algebra $(E; +, 0, 1)$ we can associate the algebra $(E; \oplus, \neg, 0)$ where the total operations $\oplus$ and $\neg$ are defined by (3.1). For example, for the diamond $(D; +, 0, 1)$ we have:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

$\neg$	0	a	b	1
	1	a	b	0

We have proved in [35] that the class of algebras that arise in this way is axiomatized by the identities

$$x \oplus 0 = x, \tag{B1}$$

$$\neg\neg x = x, \tag{B2}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \tag{B3}$$

$$\neg\big(\neg(\neg(x \oplus y) \oplus y) \oplus z\big) \oplus (x \oplus z) = 1, \tag{B4}$$

together with the quasi-identity

$$\text{(E)}\ \ x \le \neg y \quad \& \quad x \oplus y \le \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y),$$

where we put $1 = \neg 0$ and for any pair of terms we write $s \le t$ as a shorthand for $\neg s \oplus t = 1$. The quasi-identity (E) can be equivalently rewritten as an identity, e.g. as

$$\text{(E')}\ \ (x \oplus y) \oplus (\neg(x \oplus y) \wedge z) = x \oplus \big((\neg(x \oplus y) \wedge z) \oplus y\big).$$

In fact, there is a one-one correspondence between lattice effect algebras and these total algebras, just as in the case of MV-effect algebras and MV-algebras. To be more concrete, if $(E; +, 0, 1)$ is a lattice effect algebra, then $(E; \oplus, \neg, 0)$ satisfies (B1)–(B4) and (E), and the initial partial $+$ is obtained by restricting $\oplus$ to the pairs x, y with $x \le \neg y$. Conversely, if $(E; \oplus, \neg, 0)$ is an algebra satisfying (B1)–(B4) and (E), then $(E; +, 0, 1)$ is a lattice effect algebra once we put $x + y = x \oplus y$ for $x \le \neg y$; the orthosupplement of x is just $\neg x$. The quasi-identity (E) ensures that $+$ is commutative and associative in the sense of the axioms (i) and (ii).

According to [35], algebras $(E; \oplus, \neg, 0)$ of type $(2, 1, 0)$ that fulfill the above identities (B1)–(B4) are called *basic algebras*. The name 'basic algebra' was used just as a makeshift. The original motivation for the introduction of these structures was to mimic the relationship between MV-algebras and boolean algebras by replacing the latter ones with orthomodular lattices. In fact, basic algebras as such are too general from this point of view, but they are a common 'base' for other structures that could be regarded as standing to orthomodular lattices as MV-algebras stand to boolean algebras, and hence 'basic algebras'. A basic algebra is *commutative* if the operation $\oplus$ is commutative.

Basic algebras are to some extent similar to MV-algebras; especially, the stipulation

$$x \le y \quad \text{iff} \quad \neg x \oplus y = 1 \tag{3.2}$$

defines a bounded lattice in which the join $x \vee y$ and the meet $x \wedge y$ are given by

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y). \tag{3.3}$$

Moreover, for each $a \in E$, the map $\beta_a : x \mapsto \neg x \oplus a$ is an *antitone involution* on the interval $[a, 1] = \{x \in E \mid a \leq x\}$, and the algebra $(E; \oplus, \neg, 0)$ is completely determined by the lattice and the β_a's ($a \in E$), because $\neg x = \beta_0(x)$ and $x \oplus y = \beta_y(\neg x \vee y)$ for all $x, y \in E$. On the other hand, let $(B; \vee, \wedge, 0, 1)$ be a bounded lattice where every interval $[a, 1]$ is equipped with a fixed antitone involution, say γ_a. Then we can define $\neg x = \gamma_0(x)$ and $x \oplus y = \gamma_y(\neg x \vee y)$, and it is easy to verify that the algebra $(B; \oplus, \neg, 0)$ is a basic algebra; its underlying lattice is just $(B; \vee, \wedge, 0, 1)$ and the antitone involutions are the γ_a's (i.e. $\beta_a = \gamma_a$ for every $a \in B$). In other words, $(E; \oplus, \neg, 0)$ is a basic algebra if and only if

- $(E; \vee, \wedge, 0, 1)$ with $\vee$ and $\wedge$ defined by (3.3) is a bounded lattice whose induced ordering is given by (3.2),
- the maps $\beta_a : x \mapsto \neg x \oplus a$ are antitone involutions on the intervals $[a, 1]$,
- $\neg x = \beta_0(x)$ and $x \oplus y = \beta_y(\neg x \vee y)$ for all $x, y \in E$.

This approach to basic algebras as lattices with antitone involutions clarifies their arithmetical properties that we will use without explicit reference. Besides trivialities such as $y \leq x \oplus y$, $x \leq y$ iff $\neg y \leq \neg x$, $x \oplus 0 = x = 0 \oplus x$, $x \oplus 1 = 1 = 1 \oplus x$ or $\neg x \oplus x = 1 = x \oplus \neg x$, it is easy to see that the addition $\oplus$ is monotone in the first argument, i.e.

$$x \leq y \quad \Rightarrow \quad x \oplus z \leq y \oplus z. \tag{3.4}$$

Indeed, $x \leq y$ yields $\neg x \vee z \geq \neg y \vee z \geq z$ whence $\beta_z(\neg x \vee z) \leq \beta_z(\neg y \vee z)$ because β_z is an antitone involution on $[z, 1]$. On the other hand, $\oplus$ is not monotone in the second argument, i.e. $x \leq y$ need not imply $z \oplus x \leq z \oplus y$, which can be seen e.g. in the algebra $(D; \oplus, \neg, 0)$ corresponding to the diamond effect algebra $(D; +, 0, 1)$.

Having the above correspondence in mind, let $(E; +, 0, 1)$ be a lattice effect algebra. It is not hard to show that the map $\gamma_a : x \mapsto x' + a$ is an antitone involution on $[a, 1]$, for every $a \in E$. The basic algebra $(E; \oplus, \neg, 0)$ assigned to the lattice $(E; \vee, \wedge, 0, 1)$ and the antitone involutions γ_a ($a \in$

E) is defined as follows: $\neg x = \gamma_0(x) = x'$ and $x \oplus y = \gamma_y(\neg x \vee y) = (x' \vee y)' + y = (x \wedge y') + y$, i.e. exactly by (3.1), and therefore, as we have already mentioned, lattice effect algebras can be identified with basic algebras satisfying (E).

Following [32], we call a basic algebra which fulfills (E') an *effect basic algebra*. Hence the class of effect basic algebras is a variety.

Orthomodular lattices as well as MV-algebras are subvarieties of effect basic algebras: MV-algebras coincide with associative basic algebras and orthomodular lattices are term equivalent to basic algebras satisfying the identity $x \oplus (x \wedge y) = x$. It is worth observing that although the partial addition $+$ in orthomodular lattices is the restriction of $\vee$, the total addition $\oplus$ defined by (3.1), i.e. $x \oplus y = (x \wedge y') \vee y$, agrees with $\vee$ only in boolean lattices. The identity $x \oplus (x \wedge y) = x$ is a direct translation of the orthomodular law.

In effect algebras, there is an important concept of compatibility: Two elements a, b in an effect algebra $(E; +, 0, 1)$ are said to be *compatible* if there exist $a_1, b_1, c \in E$ such that $a = c + a_1$, $b = c + b_1$ and $a_1 + b_1 + c$ is defined. This is equivalent to saying that $(a \vee b)' + b = a' + (a \wedge b)$ provided that $(E; +, 0, 1)$ is a lattice effect algebra. A *block* is a maximal set of mutually compatible elements. It was proved in [35] that a, b are compatible in a lattice effect algebra if and only if $a \oplus b = b \oplus a$ in the assigned effect basic algebra, thus a block is a maximal subset B with the property that $x \oplus y = y \oplus x$ for all $x, y \in B$.

It has been known (see [137]) that any lattice effect algebra is covered by its blocks and that the blocks are MV-effect algebras, i.e., roughly speaking, lattice effect algebras are unions of MV-algebras. This can be used in order to characterize effect basic algebras within basic algebras:

> *A basic algebra is an effect basic algebra if and only if all its blocks are subalgebras that are MV-algebras.*

(More correctly, we should say that the blocks are subuniverses and the corresponding subalgebras are MV-algebras.) In the finite case one can even prove that a basic algebra is an effect basic algebra if and only if the blocks are subalgebras (cf. [35, Theorem 7.9 and Corollary 7.10]).

Consequently, in effect basic algebras, though $\oplus$ is not associative, for every integer $k \geq 1$ we can unambiguously write

$$k \cdot x = x \oplus x \oplus \cdots \oplus x$$

(with k occurrences of x) because each element together with all its 'multiples' belongs to a block, which is an MV-algebra. If we want to define $k \cdot x$ for basic algebras in general, we have to be more precise: by induction,

$$1 \cdot x = x \quad \text{and} \quad k \cdot x = ((k-1) \cdot x) \oplus x \text{ for } k > 1.$$

Once the meaning of 'k times x' is clear, we further define 'x to the k' by

$$x^k = \neg(k \cdot \neg x).$$

Since in the standard MV-algebra $([0, 1]; \oplus, \neg, 0)$ we have

$$k \cdot x = \min\{kx, 1\}$$

and

$$x^k = \max\{1 - k(1 - x), 0\},$$

the next observation is straightforward: Given a positive integer k, in all MV-chains $\mathbf{C}_n$ with $n - 1 \leq k$, since $C_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$, we have

$$k \cdot x = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad x^k = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

The same holds true in the diamond $\mathbf{D}$ provided that $k \geq 2$.

We shall write kx instead of $k \cdot x$ except for the cases where the absence of the dot could make the expressions a bit confusing.

We close this preliminary section with a few more fact about basic algebras. Like in MV-algebras, we want to define 'x minus y'. But $\oplus$ is not commutative, and so in basic algebras, there are two distinct natural subtractions defined by

$$x \ominus y = \neg(\neg x \oplus y) \quad \text{and} \quad x * y = \neg(y \oplus \neg x).$$

The addition $\oplus$ is then recovered by

$$x \oplus y = \neg(\neg x \ominus y) = \neg(\neg y * x).$$

It is obvious that $x \ominus y = \neg y * \neg x$ and $x * y = \neg y \ominus \neg x$, and the operations $\ominus$ and $*$ coincide only in commutative basic algebras. Recalling the properties of $\neg$ and $\oplus$ we easily obtain:

- $\neg x = 1 \ominus x = 1 * x$,
- $x \ominus 0 = x * 0 = x$,
- $x \leq y$ iff $x \ominus y = 0$ iff $x * y = 0$,
- $x \vee y = (x \ominus y) \oplus y$ and $x \wedge y = x * (x * y)$,
- $x \leq y \Rightarrow x \ominus z \leq y \ominus z$ and $z * y \leq z * x$,

where the last item follows from the one-sided monotonicity (3.4). Depending on the context, we alternate $\ominus$ and $*$, but we might prefer $*$ since for every a, the map $x \mapsto a * x$ is an antitone involution on the interval $[0, a]$. Hence, for $x, y \in [0, a]$ we have $a * (x \vee y) = (a * x) \wedge (a * y)$ and dually. The subtraction $\ominus$ is useful, too; for instance, for any $a \leq b$, the map $x \mapsto b \ominus x$ is a lattice isomorphism between the intervals $[a, b]$ and $[0, b \ominus a]$. Moreover, $\ominus$ is tied up with $\oplus$ via the dual residuation law $x \oplus y \geq z$ iff $x \geq z \ominus y$, thus basic algebras could be viewed as certain (dually) residuated structures.

Finally, we say that an element x in a basic algebra $\mathbf{A}$ is *sharp* if $x \wedge \neg x = 0$ (or equivalently, if $x \vee \neg x = 1$), i.e. if $\neg x$ is a complement of x in the underlying lattice. It can easily be shown that x is sharp if and only if $x \oplus x = x$ (because $\neg x \vee x = \neg(x \oplus x) \oplus x$ and $\neg x \oplus x = 1$, see [35]). In case of effect basic algebras, the sharp elements form a subalgebra (which is an orthomodular lattice in its own right), while in general, the sharp elements of a basic algebra do not even form a sublattice of the underlying lattice. Let us denote by $S(\mathbf{A})$ the set of all *sharp elements* of $\mathbf{A}$.

3.2 Finitely generated varieties of distributive effect algebras

3.2.1 Finitely generated subvarieties: Axiomatization

We identify lattice effect algebras and effect basic algebras, hence the ambient variety is the variety $\mathcal{E}$ of effect basic algebras, which is axiomatized by the identities (B1)–(B4) and (E'). $\mathcal{DE}$ is the subvariety of distributive effect basic algebras. We refer to basic algebras by boldface letters indicating the universe of the algebra, i.e. $\mathbf{E}$ is the basic algebra $(E; \oplus, \neg, 0)$. In particular, for every integer $n \geq 2$, $\mathbf{C}_n$ is the standard n-element MV-chain with the universe $C_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$, and $\mathbf{D}$ is the effect basic algebra corresponding to the diamond effect algebra $(D; +, 0, 1)$. Let us notice that all $\mathbf{C}_n$'s as well as $\mathbf{D}$ are simple algebras.

Proposition 3.1. *Up to isomorphism, every finite algebra in $\mathcal{DE}$ is of the form $\mathbf{C}_{k_1}^{m_1} \times \cdots \times \mathbf{C}_{k_s}^{m_s} \times \mathbf{D}^p$ for some integers $k_i \geq 2$, $m_i \geq 0$ and $p \geq 0$.*

Proof. This is just a reformulation of Theorem 7.9 from [95] that we have quoted in Section 3.1 because finite distributive lattice effect algebras are direct products of finite chains and diamonds, and as can easily be seen, direct products of lattice effect algebras naturally correspond to direct products of algebras in the variety $\mathcal{E}$. □

The variety of basic algebras, and hence $\mathcal{E}$ and $\mathcal{DE}$, has some nice congruence properties: besides being congruence-distributive (which follows from the fact that basic algebras are lattices), basic algebras are congruence-permutable and regular (see [35, Theorem 3.12]).

Corollary 3.1. *Every finitely generated subvariety of $\mathcal{DE}$ is generated by a finite set $\mathcal{K}$ consisting of finite MV-chains and/or $\mathbf{D}$. The subdirectly irreducible algebras in the variety $V(\mathcal{K})$ generated by $\mathcal{K}$ are again finite MV-chains and/or $\mathbf{D}$ (the subalgebras of the members of $\mathcal{K}$).*

Proof. The first statement follows from Proposition 3.1. Owing to congruence-distributivity, and since the algebras in $\mathcal{K}$ are finite and simple, the subdirectly irreducible members of $V(\mathcal{K})$ are just the subalgebras of the algebras from $\mathcal{K}$ (cf. [27, Corollary IV.6.10]). $\square$

The main goal for this section is to axiomatize all finitely generated subvarieties of $\mathcal{DE}$, i.e. the varieties $V(\mathcal{K})$ where $\mathcal{K}$ is a finite set consisting of finite MV-chains, say $\mathbf{C}_{k_1}, \dots, \mathbf{C}_{k_s}$, and the diamond $\mathbf{D}$. If $\mathcal{K}$ does not contain $\mathbf{D}$, then $V(\mathcal{K})$ is a variety of MV-algebras, and the axiomatization of finitely generated varieties of MV-algebras is well-known (see [46]). The case $\mathcal{K} = \{\mathbf{D}\}$ was solved in [32] (see below), hence we are primarily interested in the case when $\mathcal{K}$ contains MV-chains and $\mathbf{D}$.

It was proved in [32] that the variety $V(\mathbf{D})$ can be axiomatized—relative to the variety $\mathcal{DE}$—by the identity

$$(x * y) * 2z = (x * 2z) * (y * 2z).$$

(Actually, this is not the identity from [32] but its translation into the language of $\oplus$ and $*$ obtained by eventually replacing each variable v with $\neg v$.) The question naturally arises what happens if $2z$ in the identity is replaced with kz, for some integer $k > 2$? We shall prove that the identity

$$(x * y) * kz = (x * kz) * (y * kz) \tag{E$_k$}$$

axiomatizes the variety generated by the MV-chains $\mathbf{C}_2, \dots, \mathbf{C}_{k+1}$ and by the diamond $\mathbf{D}$, which is the join of $V(\mathbf{D})$ with the variety $V(\mathbf{C}_2, \dots, \mathbf{C}_{k+1})$ of k-potent MV-algebras (i.e. MV-algebras satisfying the identity $kx = (k+1)x$, see [46]).

We let $\mathcal{E}_k$ and $\mathcal{DE}_k$ denote the variety of effect basic algebras satisfying (E_k) and the variety of distributive effect basic algebras satisfying (E_k), respectively. We emphasize that in what follows k can be any positive integer, i.e. we admit that $k = 1$ or $k = 2$. It should be clear by the observation (3.5) and the properties of the subtraction $*$ that the MV-chains $\mathbf{C}_2, \dots, \mathbf{C}_{k+1}$ all belong to $\mathcal{DE}_k$, and so does $\mathbf{D}$ if $k \geq 2$.

Lemma 3.1. *Let* $\mathbf{A}$ *be a basic algebra (not necessarily an effect basic algebra) satisfying the identity* (E_k)*. Then for all* $x, y, z \in A$*:*

(i) *if* $y \le kx$, *then* $kx \oplus y = kx$;

(ii) $kx \oplus kx = kx$ *and* $kx \oplus x = kx = x \oplus kx$;

(iii) *if* $x, y \le kz$, *then* $x \oplus y \le kz$;

(iv) $x * y \le kz$ *iff* $x \ominus y \le kz$.

Proof. (i) By (E_k), for any $y \le kx$ we have $\neg y * kx = (1 * y) * kx = (1*kx)*(y*kx) = \neg kx*0 = \neg kx$ whence $kx \oplus y = \neg(\neg y*kx) = \neg\neg kx = kx$.

(ii) The equations $kx \oplus kx = kx$ and $kx \oplus x = kx$ follow directly from (i) as $x \le kx$. From the inequalities $kx \le x \oplus kx \le kx \oplus kx = kx$, which follow from (3.4), we get $x \oplus kx = kx$.

(iii) By (i) and (3.4), if $x, y \le kz$, then $x \oplus y \le kz \oplus y = kz$.

(iv) Let $x * y \le kz$. Then $0 = (x * y) * kz = (x * kz) * (y * kz)$, and so $x*kz \le y*kz$. This yields $\neg y*kz = (1*y)*kz = (1*kz)*(y*kz) \le (1*kz)*(x*kz) = (1*x)*kz = \neg x*kz$ whence $0 = (\neg y*kz)*(\neg x*kz) = (\neg y*\neg x)*kz$. Thus $x \ominus y = \neg y * \neg x \le kz$.

Since $x \ominus y = \neg y * \neg x$ and $x * y = \neg y \ominus \neg x$, the same argument proves the reverse implication: if $x \ominus y \le kz$, then by what we have just shown $\neg y * \neg x \le kz$ implies $\neg y \ominus \neg x \le kz$, i.e. $x * y \le kz$. ☐

Corollary 3.2. *The variety of basic algebras satisfying the identity* (E_1) *is term equivalent to the variety of boolean algebras.*

Proof. Let $\mathbf{A}$ be a basic algebra that fulfills (E_1). In view of Lemma 3.1 (iv), $\mathbf{A}$ satisfies the identity $x * y = x \ominus y$, which is obviously equivalent to commutativity of $\oplus$. Then $x \vee y \le x \oplus y$ for all $x, y \in A$, and Lemma 3.1 (iii) actually entails that $x \vee y = x \oplus y$. Recalling the identity $\neg x \oplus x = 1$, and since the underlying lattice of a commutative basic algebra is distributive (cf. [35]), we conclude that $(A; \vee, \wedge, \neg, 0)$ is a boolean algebra.

On the other hand, if $(B; \vee, \wedge, ', 0)$ is a boolean algebra, then the corresponding basic algebra $(B; \oplus, \neg, 0)$ satisfies (E_1) because $x \oplus y = (x \wedge y') \vee y = x \vee y$ and $x * y = \neg(y \oplus \neg x) = (y \vee x')' = x \wedge y'$, so it is evident that (E_1) is satisfied. ☐

By an *ideal* of a basic algebra we mean the 0-class of a congruence. We know that basic algebras are congruence-regular, i.e. every congruence is completely determined by its arbitrary class (see [35]). Hence, for any basic algebra $\mathbf{A}$, the map $\theta \mapsto [0]_\theta = \{a \in A \mid (a,0) \in \theta\}$ is an isomorphism between the congruence lattice of $\mathbf{A}$ and the lattice of ideals of $\mathbf{A}$. Unlike the general case, the description of ideals of effect basic algebras is surprisingly simple. Indeed, by [132], a non-empty subset $I \subseteq E$ is an ideal of an effect basic algebra $\mathbf{E}$ if and only if it satisfies the conditions:

(I1) $x \oplus y \in I$ for all $x, y \in I$,

(I2) if $x \in I$, then $x \ominus y \in I$ for every $y \in E$.

For any $a \in A$, the ideal generated by a is denoted by $\mathrm{Ig}(a)$.

Lemma 3.2. *Let* $\mathbf{E} \in \mathcal{E}_k$. *Then* $\mathrm{Ig}(a) = \mathrm{Ig}(ka) = [0, ka]$ *for every* $a \in E$.

Proof. By Lemma 3.1 (iii), the interval $[0, ka]$ satisfies (I1). It also satisfies (I2) because if $x \leq ka$, then for any $y \in E$ we have $x * y \leq x \leq ka$, which implies $x \ominus y \leq ka$ by 3.1 (iv). Thus $[0, ka]$ is an ideal of $\mathbf{E}$, whence it follows that $\mathrm{Ig}(a) = \mathrm{Ig}(ka) = [0, ka]$. □

Lemma 3.3. *A non-trivial algebra in* $\mathcal{E}_k$ *is simple if and only if it has no idempotent (= sharp) elements other than* $0, 1$. *Hence simple algebras in* $\mathcal{E}_k$ *are hereditarily simple.*

Proof. Let $\mathbf{E} \in \mathcal{E}_k$. If $a \in E \setminus \{0, 1\}$ is idempotent, then $a = ka$ yields $\mathrm{Ig}(a) = [0, a]$, and so $\{0\} \neq \mathrm{Ig}(a) \neq E$. Thus $\mathbf{E}$ is not simple. Conversely, if $\mathbf{E}$ is not simple, then there exists an ideal J such that $\{0\} \neq J \neq E$. Then for any $a \in J \setminus \{0\}$, the element ka is idempotent (by Lemma 3.1 (ii)) and $0 \neq ka \neq 1$ since $ka \in J$.

Now, if $\mathbf{E}$ is simple, then so is every a subalgebra of $\mathbf{E}$. □

Lemma 3.4. *Let* $\mathbf{E} \in \mathcal{E}_k$. *If* $\mathbf{E}$ *is subdirectly irreducible, then it is simple. Consequently,* $ka = 1$ *for all* $a \in E \setminus \{0\}$.

Proof. Let M be the monolith of the ideal lattice of $\mathbf{E}$. Then $M = \mathrm{Ig}(a)$ for some $a \in E \setminus \{0\}$; by replacing a with ka we may assume that a is idempotent, and in this case $M = [0, a]$ by Lemma 3.2. We want to show that $M = E$, i.e. $a = 1$. Suppose to the contrary that $a \neq 1$. Then $\neg a \neq 0$ and so $M \subseteq \mathrm{Ig}(\neg a)$. Recalling that an element x is idempotent if and only if $\neg x$ is a complement of x in the underlying lattice, we get that $\neg a$ is idempotent, and hence $[0, a] = M \subseteq \mathrm{Ig}(\neg a) = [0, \neg a]$, which entails $a \leq \neg a$. But then $a = a \wedge \neg a = 0$, this being the required contradiction.

The last statement now follows from the fact that each non-zero element generates a non-zero ideal, i.e. the whole E. $\square$

Lemma 3.5. *For every positive integer k, the variety $\mathcal{E}_k$ is a discriminator variety.*

Proof. By Lemma 3.4, in subdirectly irreducible members of the variety we have $ka = 0$ for $a = 0$, and $ka = 1$ otherwise. Let us define $d(x, y) = (x \ominus y) \vee (y \ominus x)$. Then $d(x, y) = 0$ iff $x = y$, and the term

$$t(x, y, z) = \big(x \ominus \neg(k \cdot d(x, y))\big) \vee \big(z \ominus (k \cdot d(x, y))\big)$$

is a discriminator term for $\mathcal{E}_k$. Indeed, for any subdirectly irreducible algebra $\mathbf{E} \in \mathcal{E}_k$ and $x, y, z \in E$, we have (i) $k \cdot d(x, x) = 0$ and $t(x, x, z) = (x \ominus \neg 0) \vee (z \ominus 0) = 0 \vee z = z$, and (ii) if $x \neq y$, then $d(x, y) \neq 0$, thus $k \cdot d(x, y) = 1$ and $t(x, y, z) = (x \ominus \neg 1) \vee (z \ominus 1) = x \vee 0 = x$. (We could have alternatively used the operations $*$ and $\oplus$ instead of $\ominus$ and $\vee$, respectively.) $\square$

Lemma 3.6. *Let $\mathbf{E}$ be a non-trivial effect basic algebra such that all its blocks are chains. If B_1, B_2 are distinct blocks of $\mathbf{E}$, then $B_1 \cap B_2 = \{0, 1\}$. Therefore $\mathbf{E}$ is the horizontal sum of its blocks.*

This needs an explanation. Suppose that $\{\mathbf{B}_i \mid i \in I\}$ is a family of non-trivial basic algebras such that $B_i \cap B_j = \{0, 1\}$ whenever $i \neq j$. Let $E = \bigcup_{i \in I} B_i$. Then by the *horizontal sum* of the $\mathbf{B}_i$'s we mean the basic algebra $\mathbf{E} = (E; \oplus_h, \neg_h, 0)$ where $x \oplus_h y = x \oplus y$ if there is $i \in I$ such that $x, y \in B_i$, and $x \oplus_h y = y$ otherwise, and $\neg_h x = \neg x$. Thus, we glue the algebras $\mathbf{B}_i$ together at 0 and 1, the addition is extended by letting $x \oplus_h y = y$ if $x, y \notin \{0, 1\}$ are not found in one algebra, and the negation is left as it is. See [35] for the details.

An example of a horizontal sum is the diamond $\mathbf{D}$. Indeed, $\mathbf{D}$ is the horizontal of two copies of $\mathbf{C}_3$: if $(\{0, a, 1\}; \oplus, \neg, 0)$ and $(\{0, b, 1\}; \oplus, \neg, 0)$ are isomorphic to $\mathbf{C}_3$, then the operations $\oplus_h$ and $\neg_h$ on $D = \{0, a, b, 1\}$ are given as follows:

$\oplus_h$	0	a	b	1
0	0	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

$\neg_h$	0	a	b	1
	1	a	b	0

Thus the horizontal sum is just $\mathbf{D}$.

Proof of the lemma. We first note that $x, y \in E$ are compatible (i.e. $x \oplus y = y \oplus x$) if and only if $x \leq y$ or $y \leq x$. Indeed, if x, y are compatible, then they belong to one block, and since by our assumption the blocks are chains, the elements x, y are comparable. Conversely, it is clear by the definition of compatibility in lattice effect algebras that comparable elements are always compatible. Thus compatible = comparable.

Suppose that there exist incomparable elements $x, y \in E$ with $x \vee y < 1$. We may additionally assume that $x \wedge y = 0$: such elements exist because for $x_0 = (x \vee y) * x$ and $y_0 = (x \vee y) * y$ we have $x_0 \vee y_0 \leq x \vee y < 1$ and $x_0 \wedge y_0 = ((x \vee y) * x) \wedge ((x \vee y) * y) = (x \vee y) * (x \vee y) = 0$. Since $x \vee y$ is compatible with both x and y, and since the blocks are subuniverses, $x \vee y$ is also compatible with $\neg x$ and $\neg y$. Thus $x \vee y \leq \neg x$ or $x \vee y \geq \neg x$. In the former case we have $y \leq \neg x$, so y is compatible with $\neg x$ and hence with x, which is impossible since compatible elements are comparable. Hence $x \vee y \geq \neg x$. The same argument gives $x \vee y \geq \neg y$. Then $x \vee y \geq \neg x \vee \neg y = \neg(x \wedge y) = \neg 0 = 1$, a contradiction. Thus, if $x, y \in E$ are incomparable, then $x \vee y = 1$, and symmetrically, $x \wedge y = 0$. The statement now follows immediately. $\square$

We now characterize the subdirectly irreducible algebras in $\mathcal{E}_k$ and $\mathcal{DE}_k$ for $k \geq 2$. By Corollary 3.2, $\mathcal{E}_1 = \mathcal{DE}_1$ is term equivalent to the variety of boolean algebras, so the only subdirectly irreducible algebra in this variety is $\mathbf{C}_2$.

Theorem 3.1. *Let $k \geq 2$. The subdirectly irreducible algebras in the variety $\mathcal{E}_k$ are the MV-chains $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}$ and their horizontal sums. The subdirectly irreducible algebras in $\mathcal{DE}_k$ are $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}$ and $\mathbf{D}$, i.e. $\mathcal{DE}_k = V(\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}, \mathbf{D})$.*

Proof. Let $\mathbf{E} \in \mathcal{E}_k$ be subdirectly irreducible. We know that every block B is a subuniverse of $\mathbf{E}$ and the subalgebra $\mathbf{B}$ is an MV-algebra. Hence, since $\mathbf{E}$ is hereditarily simple by Lemma 3.3, each $\mathbf{B}$ is a simple MV-algebra. Simple MV-algebras are (up to isomorphisms) subalgebras of the standard MV-algebra (see [46, Theorem 3.5.1]). But $\mathbf{B}$'s satisfaction of (E_k) implies that the cardinality of B is $\leq k+1$, and therefore $\mathbf{B} \cong \mathbf{C}_j$ for some $j \in \{2, \ldots, k+1\}$. Thus, by Lemma 3.6 we conclude that $\mathbf{E}$ is either an MV-chain or the horizontal sum of a family of MV-chains each of which is isomorphic to $\mathbf{C}_j$ for some $j \in \{2, \ldots, k+1\}$.

Now, let $\mathbf{E} \in \mathcal{DE}_k$ be subdirectly irreducible. Since $\mathbf{E}$ is a horizontal sum of finite MV-chains, distributivity yields that $\mathbf{E}$ is among the algebras $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}, \mathbf{D}$ (because the only non-trivial distributive horizontal sum of MV-chains is $\mathbf{D}$). □

Corollary 3.3. *For any $k \geq 2$, the variety $\mathcal{DE}_k$ is the join of the variety $V(\mathbf{C}_2, \ldots, \mathbf{C}_{k+1})$ of k-potent MV-algebras and the variety $V(\mathbf{D})$. In particular, $\mathcal{DE}_2 = V(\mathbf{D})$. Every finitely generated subvariety of $\mathcal{DE}$ is contained in the variety $\mathcal{DE}_k$, for some k.*

This does not answer the question of axiomatizing all finitely generated subvarieties of $\mathcal{DE}$. Given $\emptyset \neq \mathcal{K} \subseteq \{\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}, \mathbf{D}\}$ for some $k \geq 2$, what we know about the variety $V(\mathcal{K})$ is that (i) $V(\mathcal{K}) \subseteq \mathcal{DE}_k$ (here, k can be chosen as the maximum length of chains in the algebras in $\mathcal{K}$) and (ii) the subdirectly irreducible algebras in $\mathcal{DE}_k$ are $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}$ and $\mathbf{D}$. But we need to 'get rid of' those MV-chains among $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}$ which are not subalgebras of the algebras in $\mathcal{K}$.

Theorem 3.2. *Suppose that the set $\mathcal{K} = \{\mathbf{C}_{k_1}, \ldots, \mathbf{C}_{k_s}, \mathbf{D}\}$ is irredundant (in the sense that the variety $V(\mathcal{K})$ would be strictly smaller if any member of $\mathcal{K}$ were omitted). If $\mathcal{K} = \{\mathbf{C}_4, \mathbf{D}\}$, then $V(\mathcal{K}) = \mathcal{DE}_3$. If $\mathcal{K} \neq \{\mathbf{C}_4, \mathbf{D}\}$, let $k+1 = \max(k_1, \ldots, k_s)$. Then $k \geq 4$ and $V(\mathcal{K})$ is axiomatized—relative to $\mathcal{DE}_k$—by the identities*

$$k \cdot x^p = (p \cdot x^{p-1})^k \tag{$\mathrm{F}_{k,p}$}$$

for all $p \in \{3, \ldots, k-1\}$ that do not divide the numbers $k_1 - 1, \ldots, k_s - 1$.

Before we give the proof we should observe what is the role of the equations $(\mathrm{F}_{k,p})$ for the MV-algebra $\mathbf{C}_{k+1}$ $(k \geq 3)$. For the moment, we may assume $p \in \{2, \ldots, k-1\}$. Recalling (3.5), and since in the standard MV-algebra we have $p \cdot x = \min(px, 1)$ and $x^p = \max(1 - p(1-x), 0)$ for every $x \in [0,1]$, it follows that $k \cdot x^p = 0$ iff $x^p = 0$ iff $p(1-x) \geq 1$ iff $1 - \frac{1}{p} \geq x$, and $k \cdot x^p = 1$ otherwise. On the other hand, $(p \cdot x^{p-1})^k = 1$ iff $p \cdot x^{p-1} = 1$ iff $p(1 - (p-1)(1-x)) \geq 1$ iff $x \geq 1 - \frac{1}{p}$, and $(p \cdot x^{p-1})^k = 0$ otherwise. Thus the following are equivalent:

- $\mathbf{C}_{k+1}$ satisfies the equation $k \cdot x^p = (p \cdot x^{p-1})^k$;
- $\mathbf{C}_{k+1}$ satisfies the condition: $1 - \frac{1}{p} \geq x$ iff $1 - \frac{1}{p} > x$, for all $x \in C_{k+1}$;
- $\frac{1}{p} \notin C_{k+1} = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$;
- p does not divide k.

Proof of the theorem. The assumption that $\mathcal{K}$ is an irredundant set means that no $\mathbf{C}_{k_i}$ is a subalgebra of any other algebra in $\mathcal{K}$; in particular, $\mathbf{C}_2$ and $\mathbf{C}_3$ are not among the $\mathbf{C}_{k_i}$'s (because $\mathbf{C}_2$ and $\mathbf{C}_3$ are subalgebras of $\mathbf{D}$). Thus $k_i \geq 4$.

The case when $\mathcal{K} = \{\mathbf{C}_4, \mathbf{D}\}$ is almost obvious: we have $V(\mathcal{K}) \subseteq \mathcal{DE}_3$ and by Theorem 3.1, the subdirectly irreducible algebras in $\mathcal{DE}_3$ are $\mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ and $\mathbf{D}$, hence $\mathcal{DE}_3 \subseteq V(\mathcal{K})$.

Now, let $\mathcal{K} \neq \{\mathbf{C}_4, \mathbf{D}\}$ and $k+1 = \max(k_1, \ldots, k_s)$. We have $k \geq 4$. Further, let $\mathcal{V}$ be the subvariety of $\mathcal{DE}_k$ defined by the identities $(\mathrm{F}_{k,p})$.

Since $k \geq k_1 - 1, \dots, k_s - 1 \geq 3$, the algebras in $\mathcal{K}$ satisfy (E_k) (kz is either 0 or 1). By the previous discussion, the MV-chains $\mathbf{C}_{k_1}, \dots, \mathbf{C}_{k_s}$ satisfy the identities ($\mathrm{F}_{k,p}$) since the p's do not divide the numbers $k_1 - 1, \dots, k_s - 1$. Also $\mathbf{D}$ satisfies ($\mathrm{F}_{k,p}$) because for $p \geq 3$ we have $x^p = x^{p-1} = 1$ iff $x = 1$, and $x^p = x^{p-1} = 0$ otherwise. Hence $V(\mathcal{K}) \subseteq \mathcal{V}$.

Conversely, the subdirectly irreducible members of $\mathcal{V}$ are to be found among $\mathbf{C}_2, \dots, \mathbf{C}_{k+1}$ and $\mathbf{D}$. We must show that the satisfaction of ($\mathrm{F}_{k,p}$) eliminates those MV-chains which are not subalgebras of $\mathbf{C}_{k_1}, \dots, \mathbf{C}_{k_s}$. Suppose that $\mathbf{C}_l$ (with $l \leq k+1$) is not a subalgebra of any of the algebras $\mathbf{C}_{k_1}, \dots, \mathbf{C}_{k_s}, \mathbf{D}$. Then $l \in \{4, \dots, k\}$ and $l-1$ does not divide the numbers $k_1 - 1, \dots, k_s - 1$, hence $\mathbf{C}_l$ fulfills the identity ($\mathrm{F}_{k,p}$) for $p = l-1$. But taking $a = \frac{l-2}{l-1} = 1 - \frac{1}{l-1}$ we have $\neg a = \frac{1}{l-1}$ and $k \cdot a^{l-1} = k \cdot \neg((l-1) \cdot \neg a) = k \cdot \neg 1 = 0$ while $((l-1) \cdot a^{l-2})^k = \big((l-1) \cdot \neg((l-2) \cdot \neg a)\big)^k = ((l-1) \cdot \neg a)^k = 1^k = 1$. Therefore, the subdirectly irreducible algebras in $\mathcal{V}$ are just the subalgebras of the members of $\mathcal{K}$, which proves $\mathcal{V} = V(\mathcal{K})$. □

Let us take another look at our axiomatization of $V(\mathbf{C}_2, \dots, \mathbf{C}_{k+1}, \mathbf{D})$, viz. Theorem 3.1. When proving the lemmata that lead to Theorem 3.1, we basically relied on these two facts (see Lemma 3.1 and 3.2):

(i) for every a, the element ka is idempotent (= sharp),

(ii) $\mathrm{Ig}(a) = [0, ka]$, so if a itself is idempotent, then $\mathrm{Ig}(a) = [0, a]$.

We now show that (i) and (ii) equally hold true for *distributive* effect basic algebras satisfying the identity

$$kx = (k+1)x. \qquad (\mathrm{P}_k)$$

The property (i) easily follows from (P_k) because we can omit the brackets in $x \oplus \cdots \oplus x$, and hence we have $kx \oplus kx = kx$.

For (ii) we need one more concept from effect algebras. Let $(E; +, 0, 1)$ be a lattice effect algebra. It is known and easy to show that for any $a \in E$, the structure $([0, a]; +, 0, a)$—where the interval $[0, a]$ comes equipped with the restriction of $+$ to $[0, a]$—is a lattice effect algebra. An element

$a \in E$ is called *central* if $(E;+,0,1)$ is isomorphic to the direct product of the interval effect algebras $([0,a];+,0,a)$ and $([0,a'];+,0,a')$. By [136, Theorem 5.5], in order for $a \in E$ to be central it is necessary and sufficient that $x = (x \wedge a) \vee (x \wedge a')$ for all $x \in E$. In case of distributive lattice effect algebras this simplifies to $x \leq a \vee a'$ for all x, and this is the same as $a \vee a' = 1$ (or dually, $a \wedge a' = 0$). Hence, in distributive lattice effect algebras, central and sharp elements coincide. Translated into the language of basic algebras, for any $\mathbf{E} \in \mathcal{DE}$, the direct product decompositions correspond one-one to the idempotent (= sharp) elements, and consequently, for every idempotent $a \in E$, the interval $[0,a]$ is an ideal of $\mathbf{E}$ (because $[0,a]$ is the 0-class of a factor congruence). This is (ii).

Therefore, if we restrict ourselves to distributive effect basic algebras, then starting from Lemma 3.3 everything remains true, *mutatis mutandis*, with (P_k) in place of (E_k). That is, relative to the variety $\mathcal{DE}$,

- $V(\mathbf{C}_2,\dots,\mathbf{C}_{k+1},\mathbf{D})$ is axiomatized by the identity (P_k);
- $V(\mathbf{C}_4,\mathbf{D})$ is axiomatized by (P_3);
- if the set $\{\mathbf{C}_{k_1},\dots,\mathbf{C}_{k_s},\mathbf{D}\} \neq \{\mathbf{C}_4,\mathbf{D}\}$ is irredundant and $k+1 = \max(k_1,\dots,k_s)$, then $k \geq 4$ and $V(\mathbf{C}_{k_1},\dots,\mathbf{C}_{k_s},\mathbf{D})$ is axiomatized by (P_k) together with the identities $(\mathrm{F}_{k,p})$ for those $p \in \{3,\dots,k-1\}$ that do not divide $k_1-1,\dots,k_s-1$.

Thus, over the variety $\mathcal{DE}$, the identities (E_k) and (P_k) are equivalent to each other. But this is not the case for $\mathcal{E}$ where (P_k) is weaker; the reason is that the analogue of Lemma 3.3 fails to be true without the assumption of distributivity (in non-distributive effect basic algebras satisfying (P_k), the interval $[0,a]$ need not be an ideal even though a is idempotent).

Modular effect basic algebras also deserve a mention:

Theorem 3.3. *For $k \geq 2$, the variety $\mathcal{ME}_k$ of modular effect basic algebras that satisfy (E_k) is generated by the MV-chains $\mathbf{C}_2,\dots,\mathbf{C}_{k+1}$ and the horizontal sums of copies of $\mathbf{C}_3$.*

Proof. In view of Theorem 3.1 the subdirectly irreducibles are the MV-chains $\mathbf{C}_2, \ldots, \mathbf{C}_{k+1}$ plus their horizontal sums. But if we involved $\mathbf{C}_j$ with $j \geq 4$, then the obtained horizontal sum would not be modular. □

3.2.2 Free algebras in finitely generated subvarieties

In this part we aim at describing (for every $n \geq 1$) the free n-generator algebra $\mathbf{F}_{\mathcal{V}}(n)$ where $\mathcal{V}$ is a finitely generated subvariety of $\mathcal{DE}$. Thus $\mathcal{V} = V(\mathcal{K})$ where $\mathcal{K}$ consists of a finite number of finite MV-chains and/or $\mathbf{D}$. The variety $\mathcal{V}$ is locally finite (see [7, Corollary 1.4]): in general, if a variety is generated by a finite set of finite algebras, then it is also generated by their direct product, say $\mathbf{A}$, which is finite, and for any positive integer n, the free n-generator algebra in the variety is isomorphic to the subalgebra of the direct power $\mathbf{A}^{A^n}$ generated by the projections $\pi_i\colon (a_1, \ldots, a_n) \in A^n \mapsto a_i \in A_i$ (for $i = 1, \ldots, n$). Thus the free algebra is finite.

Now, since our variety $\mathcal{V}$ is contained in $\mathcal{DE}_k$ for some k, it is a discriminator variety (Lemma 3.5 and Corollary 3.3), and hence directly indecomposable algebras in $\mathcal{V}$ are simple (see [27, Theorem IV.9.4]). There is a more straightforward argument: by Lemma 3.3, $\mathbf{E} \in \mathcal{E}_k$ is simple exactly if it has only the trivial idempotents $0, 1$, which is equivalent to $\mathbf{E}$ being directly indecomposable, provided $\mathbf{E} \in \mathcal{DE}$.

It follows that the free algebra $\mathbf{F}_{\mathcal{V}}(n)$ is isomorphic to a direct product of simple algebras in $\mathcal{V} = V(\mathcal{K})$, i.e. subalgebras of the algebras from $\mathcal{K}$. Thus, if $\mathbf{C}_{k_1}, \ldots, \mathbf{C}_{k_s}$ and $\mathbf{D}$ are these simple algebras, then

$$\mathbf{F}_{\mathcal{V}}(n) \cong \mathbf{C}_{k_1}^{m_1} \times \cdots \times \mathbf{C}_{k_s}^{m_s} \times \mathbf{D}^p \tag{3.6}$$

and we only have to find the numbers $m_1, \ldots, m_s$ and p. We underline that $\mathbf{C}_{k_1}, \ldots, \mathbf{C}_{k_s}, \mathbf{D}$ have different meaning than in Theorem 3.2: before it was an irredundant generating set for the variety, while now it is the list (up to isomorphism) of simple algebras in $\mathcal{V}$.

Lemma 3.7. *Let $\mathbf{A} = \mathbf{A}_1^{m_1} \times \cdots \times \mathbf{A}_s^{m_s}$ where $\mathbf{A}_1, \ldots, \mathbf{A}_s$ are pairwise non-isomorphic non-trivial simple algebras in a congruence-distributive variety. Then, for every $j \in \{1, \ldots, s\}$, m_j is the number of the congruences $\theta \in$* Con$(\mathbf{A})$ *with the property that $\mathbf{A}/\theta \cong \mathbf{A}_j$.*

Proof. Since $\mathbf{A}$ belongs to a congruence-distributive variety, it has directly decomposable congruences, i.e. every congruence on $\mathbf{A}$ is of the form $\theta_1 \times \cdots \times \theta_{m_1+\cdots+m_s}$, where $\theta_1, \ldots, \theta_{m_1} \in \mathrm{Con}(\mathbf{A}_1)$, $\theta_{m_1+1}, \ldots, \theta_{m_1+m_2} \in \mathrm{Con}(\mathbf{A}_2)$, etc. But the algebras $\mathbf{A}_1, \ldots, \mathbf{A}_s$ are simple, so each $\theta_i \in \mathrm{Con}(\mathbf{A}_j)$ is either Δ_j or ∇_j (the trivial congruences on $\mathbf{A}_j$). It is therefore obvious that $\theta_1 \times \cdots \times \theta_{m_1+\cdots+m_s}$ is a maximal congruence on $\mathbf{A}$ if and only if exactly one θ_i is Δ_j, for some $j \in \{1, \ldots, s\}$.

We prove the assertion for m_1. The projections $\pi_1, \ldots, \pi_{m_1}$ of $\mathbf{A}$ onto $\mathbf{A}_1$ induce m_1 distinct (maximal) congruences $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta \cong \mathbf{A}_1$. Namely, the congruence corresponding to π_i has Δ_1 as the ith component. On the other hand, if η is a congruence on $\mathbf{A}$ such that $\mathbf{A}/\eta \cong \mathbf{A}_1$, then it is a maximal congruence since the algebra $\mathbf{A}_1$ is simple. Hence η must be one of the congruences induced by the projections $\pi_1, \ldots, \pi_{m_1}$. □

The number m_j can be determined by counting the homomorphisms of $\mathbf{A}$ onto $\mathbf{A}_j$ and the automorphisms of $\mathbf{A}_j$ (see [7]). Indeed, given a homomorphism φ of $\mathbf{A}$ onto $\mathbf{A}_j$ and an automorphism α of $\mathbf{A}_j$, the composite map $\alpha\varphi$ is a homomorphism of $\mathbf{A}$ onto $\mathbf{A}_j$, too, and both φ and $\alpha\varphi$ induce the same kernel congruence on $\mathbf{A}$. Therefore, we have

$$m_j = \frac{\text{the number of homomorphisms of } \mathbf{A} \text{ onto } \mathbf{A}_j}{\text{the number of automorphisms of } \mathbf{A}_j}.$$

Let us return to $\mathbf{F}_{\mathcal{V}}(n)$. Let $X = \{\xi_1, \ldots, \xi_n\}$ be the set of free generators. Since the maps from X to members of $\mathcal{V}$ can be uniquely lifted to homomorphisms, we are looking for the maps $\varphi\colon X \to C_{k_j}$ and $\varphi\colon X \to D$ such that the subalgebra generated by $\varphi(X)$ is all of $\mathbf{C}_{k_j}$ or $\mathbf{D}$, respectively.

The case of MV-chains is known (see [46]), we recall it here for completeness. For any MV-chain $\mathbf{C}_r$, each map $\varphi\colon X \to C_r$ is determined by the n-tuple $(l_1, \ldots, l_n) \in \{0, 1, \ldots, r-1\}^n$ where $\varphi(\xi_i) = \frac{l_i}{r-1}$. Moreover, the subalgebra of $\mathbf{C}_r$ generated by $\varphi(X) = \left\{\frac{l_1}{r-1}, \ldots, \frac{l_n}{r-1}\right\}$ is $\mathbf{C}_r$ if and only if the g.c.d. of the numbers $l_1, \ldots, l_n, r-1$ is 1. Thus, for any $\mathbf{C}_{k_j}$, the exponent m_j in (3.6) is the number of n-tuples $(l_1, \ldots, l_n) \in \{0, 1, \ldots, k_j - 1\}^n$ with the property that the g.c.d. of $l_1, \ldots, l_n, k_j - 1$ is 1.

It remains to count the maps $\varphi\colon X \to D$ such that $\varphi(X)$ generates $\mathbf{D}$. The 'bad' maps are those φ for which $\varphi(X) \subseteq \{0, a, 1\}$ or $\varphi(X) \subseteq \{0, b, 1\}$. There are $2 \cdot 3^n - 2^n$ such maps (we subtract 2^n maps $\varphi\colon X \to \{0, 1\}$). Hence the number of 'good' maps is $4^n - 2 \cdot 3^n + 2^n$. However, the algebra $\mathbf{D}$ has one non-trivial automorphism, viz. the map switching the atoms a and b. It follows that the exponent p in (3.6) is

$$2 \cdot 4^{n-1} - 3^n + 2^{n-1} = 2^{n-1} \cdot (2^n + 1) - 3^n. \tag{3.7}$$

Corollary 3.4. *Suppose that $\mathbf{D} \in \mathcal{K}$ and let $\mathcal{K}^* = (\mathcal{K} \setminus \{\mathbf{D}\}) \cup \{\mathbf{C}_3\}$. Then $V(\mathcal{K}^*)$ is a variety of MV-algebras and $\mathbf{F}_{\mathcal{V}}(n) \cong \mathbf{F}_{V(\mathcal{K}^*)}(n) \times \mathbf{D}^p$ where $p = 2^{n-1} \cdot (2^n + 1) - 3^n$. Hence, $\mathbf{F}_{\mathcal{V}}(1) = \mathbf{F}_{V(\mathcal{K}^*)}(1)$ is an MV-algebra, and if $n \geq 2$, then $\mathbf{F}_{\mathcal{V}}(n)$ is not an MV-algebra.*

Proof. The set $\mathcal{K}^*$ is $\mathcal{K}$ minus $\mathbf{D}$ plus $\mathbf{C}_3$ since $\mathbf{C}_3$ is a subalgebra of $\mathbf{D}$. The direct product $\mathbf{C}_{k_1}^{m_1} \times \cdots \times \mathbf{C}_{k_s}^{m_s}$ from (3.6) is the free n-generator algebra in $V(\mathcal{K}^*)$. If $n = 1$, then $p = 0$ by (3.7), hence $\mathbf{F}_{\mathcal{V}}(1) = \mathbf{F}_{V(\mathcal{K}^*)}(1)$. For $n \geq 2$ we have $p \geq 1$, so $\mathbf{F}_{\mathcal{V}}(n)$ cannot be an MV-algebra. $\square$

We can explicitly describe the free algebra $\mathbf{F}_{V(\mathbf{D})}(n)$:

Corollary 3.5. *For any integer $n \geq 1$,*

$$\mathbf{F}_{V(\mathbf{D})}(n) \cong \mathbf{C}_2^{2^n} \times \mathbf{C}_3^{3^n - 2^n} \times \mathbf{D}^p \cong \mathbf{F}_{V(\mathbf{C}_3)}(n) \times \mathbf{D}^p$$

where $p = 2^{n-1} \cdot (2^n + 1) - 3^n$.

Proof. Here $\mathcal{K} = \{\mathbf{D}\}$, and so $\mathcal{K}^* = \{\mathbf{C}_3\}$. The subdirectly irreducible algebras in $V(\mathbf{D})$ are $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{D}$, thus, following the notation of (3.6), we have $\mathbf{F}_{V(\mathbf{D})}(n) \cong \mathbf{C}_2^{m_1} \times \mathbf{C}_3^{m_2} \times \mathbf{D}^p$. Obviously, $m_1 = 2^n$ and $m_2 = 3^n - 2^n$ since the 'bad' maps are just $\varphi\colon X \to \{0, 1\}$, and p is given by (3.7). That $\mathbf{F}_{V(\mathbf{C}_3)}(n) \cong \mathbf{C}_2^{2^n} \times \mathbf{C}_3^{3^n - 2^n}$ follows from the fact that $\mathbf{C}_2$ and $\mathbf{C}_3$ are the only subdirectly irreducible algebras in $V(\mathbf{C}_3)$. $\square$

For example, we have: $\mathbf{F}_{V(\mathbf{D})}(1) \cong \mathbf{C}_2^2 \times \mathbf{C}_3 \cong \mathbf{F}_{V(\mathbf{C}_3)}(1)$, $\mathbf{F}_{V(\mathbf{D})}(2) \cong \mathbf{C}_2^4 \times \mathbf{C}_3^5 \times \mathbf{D} \cong \mathbf{F}_{V(\mathbf{C}_3)}(2) \times \mathbf{D}$ and $\mathbf{F}_{V(\mathbf{D})}(3) \cong \mathbf{C}_2^8 \times \mathbf{C}_3^{19} \times \mathbf{D}^9 \cong \mathbf{F}_{V(\mathbf{C}_3)}(3) \times \mathbf{D}^9$.

3.3 How to produce S-tense operators on lattice effect algebras

Introduction

Logic of quantum mechanics is an important tool for deciding and evaluating propositions and propositional formulas on a physical system describing events in microcosmos. As known in quantum physics, behaviour of these physical systems and their elements (i.e., elementary particles) differs from physical systems which are observed in classical physics and whose behaviour is ruled by the classical propositional calculus. This was the reason that Foulis and Bennett [84] introduced the so-called effect algebras describing algebraic properties of propositions on events in quantum mechanics. For a more detailed motivation, the reader is referred to the monograph [73] by Dvurečenskij and Pulmannová.

Every physical system $\mathcal{P}$ is a dynamic one which means that the true values of propositions about $\mathcal{P}$ vary in time. However, the majority of propositional calculi used for a description of $\mathcal{P}$ traditionally do not incorporate the time dimension. This means that the propositions are considered relative to a given moment. On the other hand, for a physical system $\mathcal{P}$ there are usually discerned its states.

We assume that $\mathcal{P}$ is in a state s_1 at time t_1 and it goes to a state s_2 at time t_2 (as an example may serve Minkowski spacetime $\mathbb{R}^{1,3}$ where three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional manifold for representing a spacetime – elements of $\mathbb{R}^{1,3}$ are then states in our setting). Hence this transition from the state s_1 to s_2 can be taken as a movement from time t_1 to t_2 in the time scale induced by the set S of states. Since not from every state s_1 we can realize a transition to an arbitrary state of S, there is a non-trivial binary relation R on S such that $(s_1, s_2) \in R$ if and only if the system $\mathcal{P}$ can switch from s_1 to s_2. Hence, we take advantage of thinking of the set S of states as the time scale and of the relation R as time preference, i.e., $(s_1, s_2) \in R$ if s_1 is "before" s_2 or s_2 is "after" s_1. Assuming that $(s_1, s_2) \in R$, $(s_2, s_3) \in R$ and possibly $s_1 = s_3$ means, among other things, that time loops are allowed. The pair (S, R) is then a time frame as before for tense MV-algebras. Having a time frame of a physical system $\mathcal{P}$, we can quantify our propositions in time as follows.

Let $p(t)$ be a propositional formula of a logic of the system $\mathcal{P}$ having only one free variable t which plays the role of time. We introduce two of the so-called tense operators G and H which quantify $p(t)$ over time as follows:

$G(p)(s)$ is valid if for any t with $(s, t) \in R$ the formula $p(t)$ is valid,

$H(p)(s)$ is valid if for any t with $(t, s) \in R$ the formula $p(t)$ is valid.

Thus the unary operator G is a tense operator saying "it is always going to be the case that" and H is saying "it has always been the case that".

Our first question concerns an algebraic axiomatization of tense operators. It was already studied for effect algebras in [37, 40], for Boolean algebras in [26], for MV-algebras in [54, 24] and for de Morgan algebras in [41, 81]. This axiomatization will be used also here. However, having an effect algebra $\mathbf{E}$ as an axiomatization of the propositional logic of a given physical system $\mathcal{P}$, and the tense operators G and H on $\mathbf{E}$, we can ask about the following:

(a) If a time frame is given (or it is already constructed), how can we compute tense operators G^* and H^* on $\mathbf{E}$ having at least the same values for propositional formulas $p(t)$ as by the given operators G and H.

(b) How to construct all possible pairs of tense operators on $\mathbf{E}$.

(c) Having tense operators satisfying our axiomatization on the lattice effect algebra $\mathbf{E}$ only, i.e., if a time frame is not given, under which conditions on $\mathbf{E}$ we can create the time frame in order to enable the solution of the above mentioned questions (a) and (b). This is usually called a *representation problem.*

The representation problem goes back to Rutledge (see [139]) where he studied monadic MV-algebras as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Recall that monadic MV-algebras are a particular case of tense MV-algebra (see [54]). Rutledge represented each subdirectly irreducible monadic MV-algebra as a subalgebra of a functional monadic MV-algebra.

In fact, some solutions of above questions were already found by the authors, see e.g. [40], [129] and [127]. However, the next question is how our formal and purely algebraic approach coresponds to a real physical quantum system $\mathcal{P}$.

For this sake, we prefer not to use all possible states of $\mathcal{P}$ but only the so-called Jauch-Piron states. The advantage is that these states reflect an important property of the logic of quantum mechanics, namely the so-called Jauch-Piron property saying that if the probability of propositions p and q being true is zero then there exits a proposition r covering both p and q such that the probability of r being true is also zero. In our paper, we start with lattice effect algebras for the sake of simplicity. Hence, we can take $r = p \vee q$. This is in accordance with Kolmogorovian probability theory. The study of Jauch-Piron states was motivated by the requirement that states on projection structures that qualify for having physical meaning should satisfy the natural Jauch-Piron property (see [130]).

3.3.1 S-tense operators on lattice effect algebras

In what follows, we develop and study the basic concepts of tense effect algebras and an explanation of the role of time scale. We also give a justification for considering S-tense operators, describe their construction and a framework for their representation. The main results are the following:

(a) any S-time frame (T, R) (see page 95) on a linearly ordered complete MV-algebra $\mathbf{M}$ induces S-tense operators (see page 96) on the cartesian product $\mathbf{M}^T$,

(b) conversely, any S-tense lattice effect algebra with a suitable set T of Jauch-Piron states is representable by some S-time frame (T, R) as a subalgebra of the respective cartesian product $[0, 1]^T$ equipped with the respective S-tense operators from (a).

The axiomatization of tense operators G and H in the classical propositional logic is given in [26]. For effect algebras, it was settled in [37, 40] and [129]. We can repeat the definition.

Let $\mathbf{E} = (E; +, 0, 1)$ be an effect algebra. Unary operators G and H on $\mathbf{E}$ are called *partial tense operators* if they are partial mappings of E into itself satisfying the following axioms:

(T1) $G(0) = 0$, $G(1) = 1$, $H(0) = 0$ and $H(1) = 1$

(T2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist and $H(x) \leq H(y)$ whenever $H(x), H(y)$ exist

(T3) if $x + y$ and $G(x), G(y), G(x + y)$ exist then $G(x) + G(y)$ exists and $G(x) + G(y) \leq G(x + y)$ and if $x + y$ and $H(x), H(y), H(x + y)$ exist then $H(x) + H(y)$ exists and $H(x) + H(y) \leq H(x + y)$

(T4) $x \leq GP(x)$ if $H(x')$ exists, $P(x) = H(x')'$ and $GP(x)$ exists, $x \leq HF(x)$ if $G(x')$ exists, $F(x) = G(x')'$ and $HF(x)$ exists.

If both G and H are total (i.e., G and H are mappings of E into itself defined for each $x \in E$) then G and H are called *tense operators* and the triple $(\mathbf{E}; G, H)$ is called a *tense effect algebra*.

This is exactly the case when a time frame (T, R) is not explicitly mentioned. But it may happen, if a time scale T and a relation R for this axiomatization were constructed, that the obtained binary relation R is neither reflexive nor transitive. The conditions under which R will be a quasi-order are analysed in Theorems 3.4 and 3.5.

One can immediately mention that if $\mathbf{E} = (E; +, 0, 1)$ is an effect algebra and G and H are mappings of E into itself defined by $G(1) = 1 = H(1)$ and $G(x) = 0 = H(x)$ for all $x \in E$, $x \neq 1$ then G and H are tense operators. However, these operators reveal little about the physical system $\mathcal{P}$ because all that is not identically equal to 1 is considered to be false both in the past and in the future. Hence, we are searching to find tense operators on E having maximally many values. The construction of an important class of such operators is given below in Theorem 3.6. However, it can happen that tense operators with maximally many values can loose their physical interpretation but another couple with smaller values can be more appropriate.

Since any lattice effect algebra can be covered by its maximal sub-MV-algebras called blocks [137] it is quite natural to ask that our (total) tense operators behave on MV-algebras according to the axiomatization of tense MV-algebras given by Diaconescu and Georgescu in [54]. This can be accomplished by the following axioms formulated for lattice effect algebras:

(T5) $G(x \oplus x) = G(x) \oplus G(x), H(x \oplus x) = H(x) \oplus H(x)$,

(T6) $G(x \odot x) = G(x) \odot G(x), H(x \odot x) = H(x) \odot H(x)$.

We call such tense operators G and H *E-tense operators*. Note that an *embedding* between tense effect algebras is an order reflecting morphism of effect algebras such that it commutes with the corresponding tense operators.

For any pair G and H of E-tense operators on $\mathbf{E}$ and any $x \in S(\mathbf{E})$, we have that $G(x)$ and $H(x)$ are in $S(\mathbf{E})$.

Now, in our case, by an *S-time frame* is meant a couple (T, R) where T is a non-void set and $R \subseteq T \times T$ is a reflexive and transitive relation. The last condition on R comes from the interpretion of $(s, t) \in R$ as s is before t in the non-strict sense so it would seem that R should be at least a quasi-order.

Having a lattice algebra $\mathbf{E}$ and a non-void set T, we can produce the direct power $(E^T; +, o, j)$ where the operation $+$ and the induced operations $\vee$, $\wedge$, $\oplus$, $\odot$ and $\neg$ are defined and evaluated on $x, y \in E^T$ componentwise. Moreover, o, j are such elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathbf{E}^T$ is again a lattice effect algebra.

The following theorem is an immediate corollary of Theorem 3.6 so we shall omit the proof up to Subsection 3.3.3.

Theorem 3.4. *Let* $\mathbf{M}$ *be a linearly ordered complete MV-effect algebra,* (T, R) *be an S-time frame and* G, H *be maps from* M^T *into* M^T *defined by*

$$G(x)(s) = \bigwedge\{x(t) \mid t \in T, sRt\}$$

and

$$H(x)(s) = \bigwedge\{x(t) \mid t \in T, tRs\}$$

for all $x \in M^T$ and $s \in T$. Then G and H are E-tense operators on the MV-effect algebra $\mathbf{M}^T$ such that

(T7) $G(x) \leq x, H(x) \leq x$,

(T8) $G(x) = G(G(x)), H(x) = H(H(x))$.

Motivated by Theorem 3.4 we say that E-tense operators G and H on a lattice effect algebra $\mathbf{E}$ are *S-tense* whenever they satisfy axioms (T7) and (T8) and that $(\mathbf{E}; G, H)$ is an *S-tense lattice effect algebra.*

It follows that $G(x)$ means that "x is always going to be the case – starting now" and $H(x)$ means that "x was always, and is now the case". Moreover, the condition (T8) yields that the statement "x will always be the case" is equivalent to the statement "it will always be the case that x will always be the case".

A *functional S-tense lattice effect algebra* $(\mathbf{E}; G_E, H_E)$ is an S-tense lattice effect algebra ($\mathbf{E}$ with S-tense operators G_E and H_E which is an E-effect-algebraic reduct of $[0,1]^T$ for a time frame (T, R) such that G_E is a restriction of G from Theorem 3.4 on E and H_E is a restriction of H from Theorem 3.4, respectively.

By a *functional representation of an S-tense lattice effect algebra* $\mathbf{A}$ with S-tense operators G_A and H_A, we mean simply a functional S-tense lattice effect algebra E such that there is an isomorphism of effect algebras $f : \mathbf{A} \to \mathbf{E}$ satisfying $f \circ G_A = G_E \circ f$ and $f \circ H_A = H_E \circ f$.

As mentioned before, the set of states of an effect algebra can serve as a time scale over which we can quantify propositions on a physical system $\mathcal{P}$. Hence, it is worth to know as much as possible about states on a given effect algebra. For this, we introduce particular states, named E-states and Jauch-Piron E-states, which can be used for a representation of an effect algebra into a direct product of standard MV-effect algebras on the interval $[0, 1]$. Although it can be introduced in full generality, we are concentrating only on the case of lattice effect algebras in order to reach everywhere defined mappings.

Definition 3.1. Let $\mathbf{E} = (E; +, 0, 1)$ be a lattice effect algebra. A map $s : E \to [0, 1]$ is called

1. *an E-state on* $\mathbf{E}$ if s is an E-morphism of effect algebras;
2. *a Jauch-Piron E-state on* $\mathbf{E}$ if s is an E-state and

   (JP) $s(x) = 1 = s(y)$ implies $s(x \wedge y) = 1$.

3. If there exists an order reflecting set T of E-states (Jauch-Piron E-states, respectively) on $\mathbf{E}$ then $\mathbf{E}$ is said to be *E-representable* (*E-Jauch-Piron representable*, respectively).
4. If any E-state is E-Jauch-Piron then $\mathbf{E}$ is called an *E-Jauch-Piron lattice effect algebra.*

First, note that any E-Jauch-Piron lattice effect algebra with an order reflecting set of E-states is E-Jauch-Piron representable. Also, any E-Jauch-Piron representable lattice effect algebra is E-representable.

Second, if $\mathbf{E}$ is E-representable then the induced morphism $i_E^T : \mathbf{E} \to [0, 1]^T$ (sometimes called *an embedding*) is an order reflecting morphism of effect algebras such that $i_E^T(x \oplus x) = i_E^T(x) \oplus i_E^T(x)$ and $i_E^T(x \odot x) = i_E^T(x) \odot i_E^T(x)$ for all $x \in E$.

Third, if there exists an order reflecting set T of states that are also lattice morphisms then $\mathbf{E}$ is an MV-effect algebra that is E-Jauch-Piron representable.

Fourth, if $\mathbf{E}$ is an MV-effect algebra and s a state on $\mathbf{E}$ then we always have $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$ for all $x, y \in E$. Hence in every MV-effect algebra any state s satisfies (JP). Moreover from Proposition 2.4 we know that s is an E-state if and only if s is an extremal state (MV-algebra morphism).

The next theorem gives us a solution of the representation problem for S-tense operators. It immediately follows from Theorem 3.8 so we postpone its proof until Subsection 3.3.4.

Theorem 3.5. *Let* $\mathbf{E}$ *be an E-representable E-Jauch-Piron S-tense lattice effect algebra with S-tense operators* G_E *and* H_E*. Then* $(\mathbf{E}; G_E, H_E)$ *can be embedded into the tense MV-algebra* $([0,1]^T; G, H)$ *induced by S-time frame* (T, ρ_G)*, where* T *is the set of all Jauch-Piron E-states from* E *to* $[0,1]$ *and the relation* ρ_G *is defined by*

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

In particular, the following diagram of functions commutes:

$$\begin{tikzcd} E \arrow[r, "G_E"] \arrow[d, "i_E^T"'] & E \arrow[d, "i_E^T"] \\ {[0,1]^T} \arrow[r, "G"'] & {[0,1]^T} \end{tikzcd} .$$

3.3.2 E-semi-states on lattice effect algebras

In this part we will systematically study the notion of an (Jauch-Piron) E-semi-state on a lattice effect algebra. The study of (Jauch-Piron) E-semi-states was motivated by the fact that a composition of an E-tense operator with an (Jauch-Piron) E-state is an (Jauch-Piron) E-semi-state. The basic result is Proposition 3.2 that any Jauch-Piron E-semi-state on an E-representable lattice effect algebra E is a meet of E-states.

Definition 3.2. *Let* $\mathbf{E} = (E; +, 0, 1)$ *be a lattice effect algebra. A map* $s : E \to [0,1]$ *is called*

1. an E-semi-state on $\mathbf{E}$ *if*

   (i) $s(0) = 0, s(1) = 1,$

   (ii) $s(x) + s(y) \leq s(x + y)$ *whenever* $x + y$ *is defined,*

   (iii) $s(x) \odot s(x) = s(x \odot x),$

(iv) $s(x) \oplus s(x) = s(x \oplus x)$,

2. a Jauch-Piron E-semi-state on $\mathbf{E}$ *if s is an E-semi-state and*

(JP) $s(x) = 1 = s(y)$ *implies* $s(x \wedge y) = 1$.

Lemma 3.8. *Let $\mathbf{E} = (E; +, 0, 1)$ be a lattice effect algebra, $s : E \to [0, 1]$ a Jauch-Piron E-semi-state on $\mathbf{E}$. Then s satisfies the following condition:*

$$s(x) = 1 = s(y) \text{ and } x \cdot y \text{ defined implies } s(x \cdot y) = 1.$$

Proof. Assume that $s(x) = 1 = s(y)$ and $x \cdot y$ is defined. Then there is a block $\mathbf{M}$ (see [73]) of $\mathbf{E}$ which is an MV-effect algebra containg x and y. Hence also $x \wedge y, x \odot y = x \cdot y$ and $(x \wedge y) \odot (x \wedge y)$ are in M. It follows that $s(x \wedge y) = 1$ and therefore also

$$1 = s(x \wedge y) = s(x \wedge y) \odot s(x \wedge y) = s((x \wedge y) \odot (x \wedge y)) \leq s(x \odot y) = s(x \cdot y).$$

□

Lemma 3.9. *Let $\mathbf{E} = (E; +, 0, 1)$ be a lattice effect algebra, S a non-empty set of E-semi-states on $\mathbf{E}$. Then*

(a) the pointwise meet $t = \bigwedge S$ is an E-semi-state on $\mathbf{E}$,

(b) if S is linearly ordered then $q = \bigvee S$ is an E-semi-state on $\mathbf{E}$.

Proof. (a): Let us check the conditions (i)-(iv) from Definition 3.2.

(i): Clearly, $t(0) = \bigwedge\{s(0) \mid s \in S\} = \bigwedge\{0 \mid s \in S\} = 0$ and $t(1) = \bigwedge\{s(1) \mid s \in S\} = \bigwedge\{1 \mid s \in S\} = 1$.

(ii): Assume that $x \leq y'$.There is an element $s_0 \in S$ such that $s_0(x) + s_0(y)$ is defined and clearly $t \leq s_0$. It follows that $t(x) + t(y)$ is defined. Let us compute the following

$$\begin{aligned} t(x) + t(y) &= t(x) \oplus t(y) = \bigwedge\{s_1(x) \mid s_1 \in S\} \oplus \bigwedge\{s_2(y) \mid s_2 \in S\} \\ &= \bigwedge\{s_1(x) \oplus s_2(y) \mid s_1, s_2 \in S\} \leq \bigwedge\{s(x) \oplus s(y) \mid s \in S\} \\ &= \bigwedge\{s(x) + s(y) \mid s \in S\} \leq \bigwedge\{s(x + y) \mid s \in S\} = t(x + y). \end{aligned}$$

(iii), (iv): As in Lemma 2.4, parts (iv) and (v).

(b): As above we have to verify the conditions (i)-(iv).

(i): Clearly, $q(0) = \bigvee\{s(0) \mid s \in S\} = \bigvee\{0 \mid s \in S\} = 0$ and $q(1) = \bigvee\{s(1) \mid s \in S\} = \bigvee\{1 \mid s \in S\} = 1$.

(ii): Assume that $x \leq y'$.Then, for all $u, v \in S$, $u \leq v$ we have $u(x) \leq v(x) \leq v(y)'$. This yields that $u(x) \leq \bigwedge\{w(y)' \mid w \in S, u \leq w\} = (\bigvee\{w(y) \mid w \in S, u \leq w\})' = q(y)'$. It follows that $q(x) \leq q(y)'$, i.e., $q(x) + q(y)$ is defined. Let us compute the following

$$\begin{aligned} q(x) + q(y) &= \bigvee\{s_1(x) \mid s_1 \in S\} + \bigvee\{s_2(y) \mid s_2 \in S\} \\ &= \bigvee\{s_1(x) + s_2(y) \mid s_1, s_2 \in S\} \leq \bigvee\{s(x) + s(y) \mid s \in S\} \\ &\leq \bigvee\{s(x+y) \mid s \in S\} = q(x+y). \end{aligned}$$

(iii), (iv): It follows by the same arguments as above in part (a) since $\oplus$ and $\odot$ distribute in any complete MV-algebra over arbitrary joins. $\square$

Following Corollary 2.1 we are able to compare E-semi-states s via inclusion relation looking only on the respective order filters $s^{-1}(\{1\}$.

Lemma 3.10. *Let $\mathbf{E} = (E; +, 0, 1)$ be a lattice effect algebra, s, t E-semi-states on $\mathbf{E}$. Then $t \leq s$ iff $t(x) = 1$ implies $s(x) = 1$ for all $x \in E$.*

Proof. Clearly, $t \leq s$ yields the condition $t(x) = 1$ implies $s(x) = 1$ for all $x \in E$.

Assume now that $t(x) = 1$ implies $s(x) = 1$ for all $x \in E$ is valid and that there is $y \in E$ such that $s(y) < t(y)$. Thus, there is a dyadic number $r \in (0, 1) \cap \mathbb{D}$ such that $s(y) < r < t(y)$. By Corollary 2.1 there is a term t_r in $T_{\mathbb{D}}$ such that $t_r(s(y)) < 1$ and $t_r(t(y)) = 1$. It follows that $s(t_r(y)) = t_r(s(y)) < 1$ and $t(t_r(y)) = t_r(t(y)) = 1$. The last condition yields that $s(t_r(y)) = 1$, a contradiction. $\square$

Riesz decomposition property and ideals in effect algebras

We recall that an effect algebra $\mathbf{E}$ satisfies the *Riesz Decomposition Property* ((RDP) in abbreviation) if $x \leq y_1 + y_2$ implies that there exist two elements

$x_1, x_2 \in E$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$. Any MV-algebra satisfies (RDP).

An *ideal* of an effect algebra $\mathbf{E}$ is a non-empty subset I of E such that

(i) $x \in E$, $y \in I$, $x \leq y$ imply $x \in I$,

(ii) if $x, y \in I$ and $x + y$ is defined in E, then $x + y \in I$.

A *filter* of an effect algebra $\mathbf{E}$ is an ideal in the dual effect algebra $\mathbf{E}^{op} = (E; \cdot, 1, 0)$.

We denote by $Id(\mathbf{E})$ the set of all ideals of $\mathbf{E}$. An ideal I is said to be a *Riesz ideal* if, for $x \in I$, $a, b \in E$ and $x \leq a + b$, there exist $a_1, b_1 \in I$ such that $x = a_1 + b_1$ and $a_1 \leq a$ and $b_1 \leq b$.

For example, if $\mathbf{E}$ has (RDP), then any ideal of $\mathbf{E}$ is Riesz and there is a one-to-one correspondence between congruences and ideals of $\mathbf{E}$ (see [69]), and given an ideal I, the congruence $\sim_I$ on $\mathbf{E}$ is assigned by $a \sim_I b$ iff there are $x, y \in I$ with $x \leq a$ and $y \leq b$ such that $a - x = b - y$. Then the quotient $\mathbf{E}/I$ is an effect algebra with RDP.

Moreover, if $\mathbf{E}$ is a lattice effect algebra then Riesz ideals of $\mathbf{E}$ are exactly ideals (see Section 4.1.1) of the effect basic algebra corresponding to $\mathbf{E}$.

The main supporting statement that we need (and which is essentially contained in the proof of Proposition 2.3) is

Proposition 3.2. *Let $\mathbf{E} = (E; +, 0, 1)$ be a lattice effect algebra with an order reflecting set $S = \{s : E \to [0,1] \mid \ s$ is an E-state on $\mathbf{E}\}$, t a Jauch-Piron E-semi-state on $\mathbf{E}$ and $S_t = \{s \in S \mid s \geq t\}$. Then $t = \bigwedge S_t$.*

Proof. We may assume that $E \subseteq [0,1]^S$ such that $x + y$, $x \cdot y$, $x \oplus x$ and $x \odot x$ computed in $\mathbf{E}$ gives us the same results as $x + y$, $x \cdot y$, $x \oplus x$ and $x \odot x$ computed in $[0,1]^S$ and restricted to elements from E. This means that the inclusion map $i : E \to [0,1]^S$ is an order reflecting E-morphism of lattice effect algebras. Note also that $[0,1]^S$ is a lattice effect algebra with RDP (MV-effect algebra).

Clearly, $t \leq \bigwedge S_t$. Assume that there is $x \in E$ such that $t(x) < \bigwedge S_t(x)$. Thus, there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t(x) < r < \bigwedge S_t(x)$. Again by Corollary 2.1 there is a term t_r in $T_{\mathbb{D}}$ such that $t(t_r(x)) = t_r(t(x)) < 1$. Let us put $U = \{z \in E \mid t(z) = 1\}$. The set U is by Lemma 3.8 a filter of E which is closed under finite meets and $z \in U$ yields $\mu_k(z) \in U$ for all $k \in \mathbb{N}$ since t is a Jauch-Piron E-semi-state, $t_r(x) \notin U$. Let V be a filter of $[0,1]^S$ generated by the set U. Then by [69, Proposition 3.1]

$$V = \{y \in [0,1]^S \mid \quad \exists n \in \mathbb{N}, \exists g_1, \dots, g_n \in [0,1]^S, \exists f_1, \dots, f_n \in U, \\ f_i \leq g_i, i = 1, \dots, n, y = g_1 \cdot \ \dots \ \cdot g_n\}.$$

Assume that $t_r(x) \in V$. Then $\exists n \in \mathbb{N}, \exists g_1, \dots, g_n \in [0,1]^S, \exists f_1, \dots, f_n \in U, f_i \leq g_i, i = 1, \dots, n, t_r(x) = g_1 \cdot \ \dots \ \cdot g_n$. We then put $f = f_1 \wedge \dots \wedge f_n$. Let $k \in \mathbb{N}$ be minimal such that $n \leq 2^k$. It follows that

$$t_r(x) = g_1 \cdot \ \dots \ \cdot g_n \cdot \underbrace{1 \cdot \ \dots \ \cdot 1}_{2^k - n \text{ times}} \geq \mu_k(f) \in U,$$

a contradiction.

So we have that $t_r(x) \notin V$. Let W be a maximal filter of $[0,1]^S$ which does contain V and does not contain $t_r(x)$. Then the set $I = \{y \in [0,1]^S \mid y' \in W\}$ is a prime ideal in $[0,1]^S$, $t_r(x)' \notin I$. It follows by [69, Proposition 6.5 and Proposition 6.10] that $[0,1]^S/I$ is a linearly ordered effect algebra, i.e. an MV-algebra such that $t_r([x]_I) = [t_r(x)]_I \neq [1]_I$. In particular the factor map $\pi_I : [0,1]^S \to [0,1]^S/I$ is an MV-morphism such that $\pi_I(x) = [x]_I$ and $\pi_I(U) \subseteq \pi_I(V) \subseteq \pi_I(W) = \{[1]_I\}$.

Let us denote by U_I the maximal ideal of $[0,1]^S/I$ and by $\overline{s} : [0,1]^S/I \to [0,1]$ the corresponding MV-morphism. Hence by Proposition 2.1 $\overline{s}([x]_I) < r < 1$. Let us put $s = \overline{s} \circ \pi_I \circ i$. Then s is an E-state such that $s(U) = \overline{s}(\pi_I(U)) = \overline{s}(\{[1]_I\}) = \{1\}$. It follows by Lemma 3.10 that $t \leq s$, i.e., $s \in S_t$ and $s(x) = \overline{s}([x]_I) < r < \bigwedge S_t(x) \leq s(x)$, a contradiction. $\square$

3.3.3 Functions between lattice effect algebras and their construction

This part studies the notion of an EM-function between lattice effect algebras and a very Jauch-Piron EM-function between lattice effect algebras. The overall goal of this section is to establish in some sense a canonical construction of very Jauch-Piron EM-functions. We would like to understand this construction because its particular case is the construction of S-tense operators when we apply the time frame (T, R). This construction can serve as an ultimate source of numerous examples.

Definition 3.3.

1. By an *EM-function G between lattice effect algebras* is meant a function $G : E_1 \to E_2$ such that $\mathbf{E}_1 = (E_1; +_1, 0_1, 1_1)$ and $boE_2 = (E_2; +_2, 0_2, 1_2)$ are lattice effect algebras and

   (EM1) $G(0_1) = 0_2$, $G(1_1) = 1_2$,

   (EM2) $G(x) +_2 G(y) \leq G(x +_1 y)$,

   (EM3) $G(x) \odot_2 G(x) = G(x \odot_1 x)$,

   (EM4) $G(x) \oplus_2 G(x) = G(x \oplus_1 x)$.

   If $E_1 = E_2$ we say that G is an *EM-operator* on E_1.

2. If moreover G satisfies conditions

   (EM2)' $G(x) +_2 G(y) = G(x +_1 y)$

   we say that G is an *EM-morphism between lattice effect algebras.*

3. If G is an EM-function between lattice effect algebras such that (EM5) (resp. (EM5)') is satisfied we say that G is *Jauch-Piron* (resp. *very Jauch-Piron*)

   (EM5) $G(x) = 1_2 = G(y)$ implies $G(x \wedge_1 y) = 1_2$,

   (EM5)' $G(x) \wedge_2 G(y) = G(x \wedge_1 y)$.

4. If G is an EM-function between lattice effect algebras such that

   (EM6) $G(x) \odot_2 G(y) \leq G(x \odot_1 y)$,

   (EM7) $G(x) \oplus_2 G(y) \leq G(x \oplus_1 y)$,

   (EM8) $G(x^n) = G(x)^n$ for all $n \in \mathbb{N}$,

   (EM9) $n \times_2 G(x) = G(n \times_1 x)$ for all $n \in \mathbb{N}$,

   we say that G is an *FEM-function.*

5. If $G : E_1 \to E_2$ and $H : B_1 \to B_2$ are EM-functions between lattice effect algebras, then a *morphism between G and H* is a pair (φ, ψ) of morphisms of lattice algebras $\varphi : E_1 \to B_1$ and $\psi : E_2 \to B_2$ such that $\psi(G(x)) = H(\varphi(x))$, for any $x \in E_1$.

Note that (EM2)' yields (EM2), (EM5)' yields (EM5). By essentially same considerations as in Lemma 3.8 the condition (EM5) yields the following:

$$G(x) = 1_2 = G(y) \text{ and } x \cdot y \text{ defined implies } G(x \odot_1 y) = 1_2.$$

Also, a composition of EM-functions (very Jauch-Piron EM-functions, FEM-functions) is an EM-function (a very Jauch-Piron EM-function, an FEM-function) again and a composition of a very Jauch-Piron EM-function with a Jauch-Piron EM-function is a Jauch-Piron EM-function.

The notion of an EM-function generalizes both the notions of an E-semistate, of a $\odot$-operator from [127] for MV-algebras and of an fm-function.

According to both (EM3) and (EM4), $G|_{S(\mathbf{E}_1)} : S(\mathbf{E}_1) \to S(\mathbf{E}_2)$ is an EM-function (a Jauch-Piron EM-function, a very Jauch-Piron EM-function) whenever G has the corresponding property.

Recall also, that EM-functions between E-representable MV-effect algebras (which are exactly semisimple MV-algebras) coincide with FEM-functions between them (see Proposition 2.6). On the contrary, there is an MV-effect algebra $\mathbf{M}$ with an EM-operator on $\mathbf{M}$ that is not an FEM-operator (see Proposition 2.7).

Lemma 3.11. *Let $G : E_1 \to E_2$ be an EM-function between lattice effect algebras, $r \in (0,1) \cap \mathbb{D}$. Then $t_r(G(x)) = G(t_r(x))$ for all $x \in E_1$.*

Proof. Note that $G(x) \oplus_2 G(x) = G(x \oplus_1 x)$ by (EM4) and $G(x) \odot_2 G(x) = G(x \odot_1 x)$ by (EM3). Then, since $t_r \in T_{\mathbb{D}}$ is defined inductively using only the operations $(-) \oplus (-)$ and $(-) \odot (-)$, we get $t_r(G(x)) = G(t_r(x))$. □

An *E-frame* is a frame (S, T, R) satisfying that for every $x \in S$ there is $y \in T$ such that xRy and for every $y \in T$ there is $x \in S$ such that xRy. If $S = T$, we will write briefly (T, R) for the E-frame (T, T, R) and we speak about an *E-time frame.* Having a lattice algebra $(E; +, 0, 1)$ and a non-void set T, we can produce the direct power $(E^T; +, o, j)$ where the operation $+$ and the induced operations $\vee$, $\wedge$, $\oplus$, $\odot$ and $\neg$ are defined and evaluated on $p, q \in E^T$ componentwise. Moreover, o, j are such elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathbf{E}^T$ is again a lattice effect algebra.

The notion of E-frame allows us to construct examples of FEM-functions between lattice effect algebras.

In what follows we will need the following

Lemma 3.12. ([37, 40]) *Let $\mathbf{E} = (E; +, 0, 1)$ be an effect algebra. Let $a_i, b_i, c_i \in E$ for $i \in I$ and assume $a_i \perp b_i$ for all $i \in I$. Let $\bigwedge\{a_i \mid i \in I\}$, $\bigwedge\{b_i \mid i \in I\}$, $\bigwedge\{c_i \mid i \in I\}$, $\bigwedge\{c_i' \mid i \in I\}$ and $\bigwedge\{a_i + b_i \mid i \in I\}$ exist. Then*

(1) $\bigwedge\{a_i \mid i \in I\} + \bigwedge\{b_i \mid i \in I\}$ exists and

$$\bigwedge\{a_i \mid i \in I\} + \bigwedge\{b_i \mid i \in I\} \leq \bigwedge\{a_i + b_i ; i \in I\}$$

(2) $\bigwedge\{c_i' \mid i \in I\} \leq (\bigwedge\{c_i \mid i \in I\})'$.

We now prove a generalization of Theorem 2.17.

Theorem 3.6. *Let $\mathbf{M}$ be a linearly ordered complete MV-effect algebra, (S, T, R) be an E-frame and G be a map from M^T into M^S defined by*

$$G(p)(s) = \bigwedge\{p(t) \mid t \in T, sRt\},$$

for all $p \in M^T$ and $s \in S$. Then G is a very Jauch-Piron FEM-function between MV-effect algebras which has a left adjoint P, i.e., $P(q) \le p$ iff $q \le G(p)$ for all $q \in M^S$ and $p \in M^T$. In this case, for all $q \in M^S$ and $t \in T$,

$$P(q)(t) = \bigvee\{q(s) \mid s \in T, sRt\}$$

and $P : (M^S)^{op} \to (M^T)^{op}$ is a very Jauch-Piron FEM-function between MV-algebras.

Proof. Trivially we can verify $G(o)(s) = 0, G(j)(s) = 1$ for all $s \in S$ due to the fact that $o(t) = 0$ and $j(t) = 1$ for each $t \in T$ thus (EM1) holds. Let us check (EM2). Assume that $p, q \in M^T$ and $p + q$ exists. Hence, $p(t) + q(t)$ exists for each $t \in T$. Let $s \in S$. By Lemma 3.12 (1) also $\bigwedge\{p(t); sRt\} + \bigwedge\{q(t); sRt\}$ exists and $G(p)(s)+G(q)(s) = \bigwedge\{p(t); sRt\}+\bigwedge\{q(t); sRt\} \le \bigwedge\{p(t) + q(t); sRt\} = G(p + q)(s)$. Thus $G(p) + G(q) \le G(p + q)$.

The conditions (EM5)', (EM6)-(EM9) and the adjointness between P and G follow directly from Theorem 2.16, the condition (EM3) follows from (EM8) and the condition (EM4) follows from (EM9). The remaining part for P follows by the same arguments applied to the dual MV-effect algebra $\mathbf{M}^{op}$. □

We say that $G : M^T \to M^S$ is the *canonical very Jauch-Piron FEM-function induced by the E-frame (S, T, R) and the MV-algebra* $\mathbf{M}$.

Corollary 3.6. *Let $\mathbf{M}$ be a linearly ordered complete MV-effect algebra, (S, R) be an E-frame, G and H be maps from M^S into M^S defined by*

$$\begin{aligned} G(p)(s) &= \bigwedge\{p(t) \mid t \in S, sRt\}, \\ H(p)(s) &= \bigwedge\{p(t) \mid t \in S, tRs\} \end{aligned}$$

for all $p \in M^S$ and $s \in S$. Then G (H) is a very Jauch-Piron E-tense operator on M^S which has a left adjoint P (F) and $(M^S; G, H)$ is a tense effect algebra. In this case, for all $q \in M^S$ and $t \in S$,

$$\begin{aligned} P(q)(t) &= \bigvee\{q(s) \mid s \in S, sRt\}, \\ F(q)(t) &= \bigvee\{q(s) \mid s \in S, tRs\}. \end{aligned}$$

Now we give the postponed proof of Theorem 3.4. It is enough to check conditions (T7) and (T8), the remaining parts follow immediately from Corollary 3.6. But (T7) and (T8) follow from Theorem 2.21.

3.3.4 The representation theorem and its applications

The aim of this part is to show that an E-representable E-Jauch-Piron lattice effect algebra with E-tense operators G and H (or with a very Jauch-Piron operator G) can be represented in a power of the standard MV-algebra $[0,1]$, where the set of all Jauch-Piron E-states serves as a E-time frame (in the sense given in Introduction) with a relation defined by means of the point-wise ordering of these states on x as well as on $G(x)$ or $H(x)$. It properly means that for every E-representable E-Jauch-Piron lattice effect algebra with tense operators a suitable E-time frame exists and can be constructed by use of the previously introduced concepts.

Theorem 3.7. *Let $G : E_1 \to E_2$ be a very Jauch-Piron EM-function between lattice effect algebras. Let $\mathbf{E}_1$ be an E-representable lattice effect algebra and let $\mathbf{E}_2$ be an E-Jauch-Piron representable lattice effect algebra, T a set of all E-states from $\mathbf{E}_1$ to the standard MV-effect algebra $[0,1]$ and S a set of all Jauch-Piron E-states from $\mathbf{E}_2$ to $[0,1]$.*

Further, let (S,T,ρ_G) be a frame such that the relation $\rho_G \subseteq S \times T$ is defined by

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E_1.$$

Then G is representable via the canonical very Jauch-Piron FEM-function $G^* : [0,1]^T \to [0,1]^S$ between MV-effect algebras induced by the E-frame (S,T,ρ_G) and the standard MV-effect algebra $[0,1]$, i.e., the following diagram of EM-functions commutes:

$$\begin{tikzcd} E_1 \arrow[r, "G"] \arrow[d, "i^T_{\mathbf{E}_1}"'] & E_2 \arrow[d, "i^S_{\mathbf{E}_2}"] \\ {[0,1]^T} \arrow[r, "G^*"'] & {[0,1]^S} \end{tikzcd} .$$

Proof. Assume that $x \in E_1$ and $s \in S$. Then $i^S_{\mathbf{E}_2}(G(x))(s) = s(G(x)) \leq t(x)$ for all $t \in T$ such that $(s,t) \in \rho_G$. It follows that $i^S_{\mathbf{E}_2}(G(x)) \leq G^*(i^T_{\mathbf{E}_1}(x))$.

Note that $s \circ G$ is a Jauch-Piron semi-state on $\mathbf{E}_1$ and by Proposition 3.2 we get that

$$\begin{aligned} s \circ G &= \bigwedge\{t : E_1 \to [0,1] \mid \ t \text{ is an E-state}, t \geq s \circ G\} \\ &= \bigwedge\{t \in T \mid \ (s,t) \in \rho_G\}. \end{aligned}$$

This yields that actually $i^S_{\mathbf{E_2}}(G(x)) = G^*(i^T_{\mathbf{E}_1}(x))$. □

Note also that Theorem 2.18 is a particular case of Theorem 3.6 for fm-functions G such that $G(0) = 0$.

The following result which is an immediate corollary of Theorem 3.7 generalizes the main result of the paper [127, Theorem 4.5].

Corollary 3.7. (Representation theorem for lattice effect algebras with a very Jauch-Piron operator) *For any E-representable E-Jauch-Piron lattice effect algebra* $\mathbf{E}$ *with a very Jauch-Piron operator* G *and where* T *is an order reflecting set of all Jauch-Piron E-states,* $\mathbf{E}$ *is embeddable via E-morphism* $i^T_{\mathbf{E}}$ *into the canonical MV-algebra* $\mathbf{L}_G = ([0,1]^T; G^*)$ *with a very Jauch-Piron operator* G^* *induced by the canonical frame* (T, ρ_G) *and the standard MV-effect algebra* $[0,1]$*. Further, for all* $x \in E$ *and for all* $s \in T$*,* $s(G(x)) = G^*((t(x))_{t \in T})(s)$*.*

The next theorem yields a solution of the representation problem for E-tense operators. The idea of the proof is taken from Theorem 2.20.

Theorem 3.8. *Let* $\mathbf{E}$ *be an E-representable E-Jauch-Piron lattice effect algebra with E-tense operators* G *and* H*. Then* $(\mathbf{E}; G, H)$ *can be embedded into the tense MV-algebra* $([0,1]^T; G^*, H^*)$ *induced by the frame* (T, ρ_G)*, where* T *is the set of all Jauch-Piron E-states from* $\mathbf{E}$ *to* $[0,1]$ *and the relation* ρ_G *is defined by*

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

Proof. We may assume that $E \subseteq [0,1]^T$. First, let us define a second relation $\rho_H \subseteq T^2$ by the stipulation:

$$t\rho_H s \text{ if and only if } t(H(x)) \leq s(x) \text{ for any } x \in E.$$

We show that the equality $\rho_G = \rho_H^{-1}$ holds. Let us suppose that $s\rho_G t$ for some $s, t \in T$. Due to the definition of E-tense operators we have $G(H(x)')' \leq x$ and hence $x' \leq G(H(x)')$, i.e., $s(x') \leq s(G(H(x)'))$. Then $s\rho_G t$ yields $s(G(H(x)')) \leq t(H(x)')$ and together with the preceding we get $s(x') \leq t(H(x)')$. It follows that $t(H(x)) \leq s(x)$ for any $x \in E$.

Due to the definition of ρ_H we have $t\rho_H s$ and $\rho_G \subseteq \rho_H^{-1}$. Analogously we can prove the second inclusion.

The remaining part follows from Theorem 3.7. Basically, the obtained equations $G^*(x) = G(x)$ and $H^*(x) = H(x)$ finish the proof. □

Now we give the postponed proof of Theorem 3.5. It is enough to show that the relation ρ_G from Theorem 3.8 is reflexive and transitive. But to prove this we can use the same arguments as in Theorem 2.21 or [129, Lemma 4.7].

The next example shows an E-representable E-Jauch-Piron lattice effect algebra $\mathbf{E}$, that is not an MV-effect algebra, with all possible (E- and S-) tense operators. In particular, we constructed a (time) frame (T, ρ) from the set T of all Jauch-Piron E-states on E and we also described a respective representation in $[0,1]^T$, in accordance to Theorem 3.8.

Example 3.1. Let us consider a lattice effect algebra $\mathbf{E} = (E; +, 0, 1)$, where $E = \{0, a, b, c, a + b = 2c = 1\}$ (see Figure 3.1). It is a horizontal

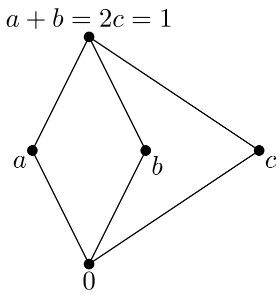

Figure 3.1: $(E;+,0,1)$

sum of a Boolean algebra $\{0,a,b,a+b=1\}$ and an MV-chain $\{0,c,2c=1\}$. Let us have pairs of tense operators G_i, H_i on $\mathbf{E}$ for some $i \in I \subseteq \mathbb{N}$. By definition we have $G_i(1) = H_i(1) = 1$ and $G_i(0) = H_i(0) = 0$ for any $i \in I$. For every $i \in I$, it holds $G_i(c) \in \{0,c\}$. Moreover, if $G_i(c) = c$ then $H_i(c) = c$ and $G_i(c) = 0$ implies $H_i(c) = 0$.

$$G_1 = H_1 = \mathrm{id}_E,$$

$$G_2(a) = a, G_2(b) = 0, G_2(c) = c,\ H_2(a) = 0, H_2(b) = b, H_2(c) = c,$$

$$G_3(a) = 0, G_3(b) = b, G_3(c) = c,\ H_3(a) = a, H_3(b) = 0, H_3(c) = c,$$

$$G_4(a) = 0, G_7(b) = 0, G_7(c) = c,\ H_7(a) = 0, H_7(b) = 0, H_7(c) = c$$

$$G_5(a) = b, G_4(b) = a, G_4(c) = c,\ H_4(a) = b, H_4(b) = a, H_4(c) = c,$$

$$G_6(a) = b, G_5(b) = 0, G_5(c) = c,\ H_5(a) = b, H_5(b) = 0, H_5(c) = c,$$

$$G_7(a) = 0, G_6(b) = a, G_6(c) = c,\ H_6(a) = 0, H_6(b) = a, H_6(c) = c,$$

Additional pairs of tense operators G_{j+7}, H_{j+7} can be defined by setting $G_{j+7}(c) = H_{j+7}(c) = 0$ and $G_{j+7}(a) = G_j(a), H_{j+7}(a) = H_j(a), G_{j+7}(b) = G_j(b), H_{j+7}(b) = H_j(b)$ for any $j \in \{1,\ldots,7\}$. From equalities $a \oplus a = a$, $b \oplus b = b$ and $c \oplus c = c + c = 1$ it follows, that the properties (T5) and (T6) are satisfied if and only if $G_i(c) = H_i(c) = c$, hence E-tense operators are only $G_1, \dots G_7, H_1 \dots H_7$. The axioms (T7) and (T8), which are necessary for E-tense operators to be S-tense, hold only for $G_1, \dots G_4, H_1 \dots H_4$.

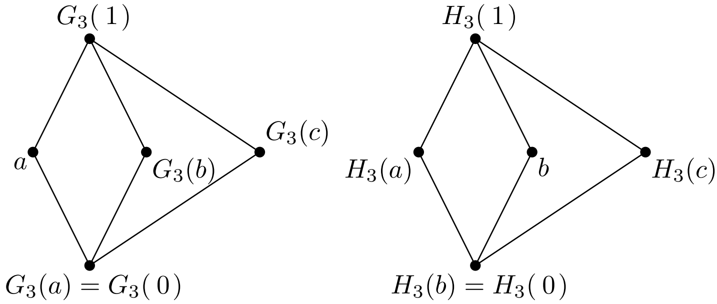

Figure 3.2: Example of S-tense operators G_3, H_3.

There exist two extremal states $T = \{s_1, s_2\}$, given by $s_1(a) = 1, s_1(b) = 0$, $s_2(a) = 0, s_2(b) = 1$ and $s_1(c) = s_2(c) = \frac{1}{2}$. The set T is also the set of all E-states, it is order reflecting and s_1, s_2 are Jauch-Piron, hence $\mathbf{E}$ is an E-representable E-Jauch-Piron lattice effect algebra.

By Theorem 3.8 for every E-representable E-Jauch-Piron lattice effect algebra $(\mathbf{E}; G_i, H_i)$ with E-tense operators there exists an embedding into a tense MV-algebra $([0,1]^T, G_i^*, H_i^*)$ with a frame (T, ρ_{G_i}), where relation ρ_{G_i} is given by the following:

$\rho_{G_1} : s_1\rho_{G_1}s_1,\ s_2\rho_{G_1}s_2,$

$\rho_{G_2} : s_1\rho_{G_2}s_1,\ s_2\rho_{G_2}s_1,\ s_2\rho_{G_2}s_2,$

$\rho_{G_3} : s_1\rho_{G_3}s_1,\ s_1\rho_{G_3}s_2,\ s_2\rho_{G_3}s_2,$

$\rho_{G_4} : s_1\rho_{G_7}s_1,\ s_2\rho_{G_4}s_1,\ s_2\rho_{G_7}s_1,\ s_2\rho_{G_7}s_2.$

$\rho_{G_5} : s_1\rho_{G_4}s_2,\ s_2\rho_{G_5}s_1,$

$\rho_{G_6} : s_1\rho_{G_5}s_1,\ s_1\rho_{G_6}s_2,\ s_2\rho_{G_5}s_1,$

$\rho_{G_7}: s_2\rho_{G_6}s_2,\ s_1\rho_{G_7}s_2,\ s_2\rho_{G_6}s_1,$

We can see that relation ρ_i is reflexive and transitive iff $i \in \{1,\ldots,4\}$, that is for the case when G_i, H_i are S-tense operators. The embedding of $\mathbf{E}$ into $[0,1]^T$ is then given by $i_E^T(a) = (1,0)$, $i_E^T(b) = (0,1)$ and $i_E^T(c) = (\frac{1}{2}, \frac{1}{2})$. As an example, let us investigate G_3^* and H_3^*.

$G_3^*(i_E^T(a)) = G_3^*(1,0) = (\bigwedge\{t(a) \mid t \in \{s_1,s_2\}, s_1\rho_3 t\}, \bigwedge\{t(a) \mid t \in \{s_1,s_2\}, s_2\rho_3 t\}) = (s_2(a), s_2(a)) = (0,0),$

$G_3^*(i_E^T(b)) = G_3^*(0,1) = (\bigwedge\{t(b) \mid t \in \{s_1,s_2\}, s_1\rho_3 t\}, \bigwedge\{t(b) \mid t \in \{s_1,s_2\}, s_2\rho_3 t\}) = (s_1(b), s_2(b)) = (0,1),$

$G_3^*(i_E^T(c)) = G_3^*(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}),$

$H_3^*(i_E^T(a)) = H_3^*(1,0) = (\bigwedge\{t(a) \mid t \in \{s_1,s_2\}, t\rho_3 s_1\}, \bigwedge\{t(a) \mid t \in \{s_1,s_2\}, t\rho_3 s_2\}) = (s_1(a), s_2(a)) = (1,0),$

$H_3^*(i_E^T(b)) = H_3^*(0,1) = (\bigwedge\{t(b) \mid t \in \{s_1,s_2\}, t\rho_3 s_1\}, \bigwedge\{t(b) \mid t \in \{s_1,s_2\}, t\rho_3 s_2\}) = (s_1(b), s_1(b)) = (0,0),$

$H_3^*(i_E^T(c)) = H_3^*(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}).$

There can be defined additional nine relations on the set T, namely $\mathcal{R} = \{\emptyset, \{[s_1,s_1]\}, \{[s_1,s_2]\}, \{[s_2,s_1]\}, \{[s_2,s_2]\}, \{[s_1,s_1],[s_1,s_2]\}, \{[s_2,s_2],[s_2,s_1]\}, \{[s_1,s_1],[s_2,s_1]\}, \{[s_2,s_2],[s_1,s_2]\}\}$. For any relation $\delta \in \mathcal{R}$, the maps G_δ^*, H_δ^* which are defined as in Corollary 3.6 by the prescriptions

$$\begin{aligned} G_\delta^*(p)(s) &= \bigwedge\{p(t) \mid t \in T, s\delta t\}, \\ H_\delta^*(p)(s) &= \bigwedge\{p(t) \mid t \in T, t\delta s\} \end{aligned}$$

for all $p \in [0,1]^T$ and $s \in T$ will not satisfy the axiom (T1). Hence G_δ^*, H_δ^* are not a pair of tense operators on $[0,1]^T$. Moreover, for any $\sigma \in \mathcal{R}, \sigma \neq \emptyset$, it holds $G_\sigma^*(i_E^T(c)) = G_\sigma^*(\frac{1}{2}, \frac{1}{2}) \notin i_E^T(E)$ or $H_\sigma^*(\frac{1}{2}, \frac{1}{2}) \notin i_E^T(E)$, $i_E^T(E) \subseteq [0,1]^T$, i.e., G_σ^*, H_σ^* define only partial (but not partial tense) operators G_σ, H_σ on E.

Chapter 4

Non-associative logics

Lambek [115], 50 years ago, considered a non-associative variant of his famous syntactic calculus (see [114], [115]). We can also find numerous papers arguing that one should not always hold on to associativity because, as pointed out in [83], "when one works with binary conjunctions and there is no need to extend them for three or more arguments (...) associativity of the conjunction is an unnecessarily restrictive condition." Accordingly, in fuzzy logics where the truth values are taken from the real interval $[0, 1]$, some authors suggest that t-norms should be replaced with copulas or quasicopulas (see [63], [98]) or even with functions satisfying certain weaker assumptions such as conjunctors or semicopulas (in our framework we have left-continuous semicopulas, see [63]).

Non-associative functions are considered also in the context of statistics, see e.g. [105], [123] or [124]. In the probabilistic framework the dependence structure of a bivariate random vectors, which are in general non-associative functions. Their connections to fuzzy logic were studied e.g. in [63] or [98], where many important results can be found.

Non-associative functions, called disjunctors, have been succesfully used for modelling the multi-valued implication operator in fuzzy logic, see [146]. Note that conjunctors and disjunctors are special so-called aggregation operators, including also copulas and co-copulas. While t-norms and tconorms require commutativity and associativity, these properties are not required by copulas, co-copulas, conjunctors and disjunctors.

Hence it seems to be important to search for some reasonable generalizations of fuzzy logics (e.g. Łukasiewicz, Gödel or product logic) having a *non-associative* conjunction.

4.1 Commutative basic algebras

Likewise we skip associativity and attempt to establish the notion of state in the fuzzy logic $\mathscr{L}_{CBA}$ which is quite close to the Łukasiewicz logic, but differs in that the conjunction is non-associative. This logic has recently been introduced in [22] as the propositional logic with the axioms

$$\varphi \to (\psi \to \varphi), \tag{Ax1}$$

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi), \tag{Ax2}$$

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi), \tag{Ax3}$$

$$\neg\neg\varphi \to \varphi, \tag{Ax4}$$

$$(\varphi \to \psi) \to (\varphi \to \neg\neg\psi), \tag{Ax5}$$

and modus ponens and the rule of suffixing

$$\text{(MP)} \quad \frac{\varphi, \varphi \to \psi}{\psi} \qquad\qquad \text{(Sf)} \quad \frac{\varphi \to \psi}{(\psi \to \chi) \to (\varphi \to \chi)}$$

as inference rules. The conjunction $\varphi \,\&\, \psi$ is definable as $\neg(\varphi \to \neg\psi)$. Commutativity of $\&$ is provable, but associativity is not; as a matter of fact, associativity of $\&$ would make $\mathscr{L}_{CBA}$ the Łukasiewicz logic. Also, if we replace (Ax4) and (Ax5) with the axiom $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ and skip the rule (Sf), which is then supefluous, we have just the Łukasiewicz logic.

$\mathscr{L}_{CBA}$ is an algebraizable logic in the sense of [11] and its equivalent algebraic sematics is the variety of *commutative basic algebras* (see [22]). The name "basic algebra" first appears in the paper [35] where we have primarily aimed at describing the class of algebras $(A; \oplus, \neg, 0)$ of type $(2, 1, 0)$

that stand to orthomodular lattices as MV-algebras stand to Boolean algebras, i.e., we wanted the idempotent elements to form the "orthomodular skeleton" of $(A; \oplus, \neg, 0)$. The simplest situation considered in [35] was the case when the underlying poset of $(A; \oplus, \neg, 0)$ defined as in MV-algebras is a bounded lattice whose *sections* (= intervals $[a, 1]$) are equipped with antitone involutions; such algebras are therefore called *basic algebras*[a] (for the exact definition see below), and *commutative basic algebras* are those with commutative $\oplus$. Among others, examples of commutative basic algebras include MV-algebras (= commutative basic algebras with associative $\oplus$) and Boolean algebras (= commutative basic algebras with idempotent $\oplus$). Hence we may regard commutative basic algebras as a non-associative generalization of MV-algebras. It is worth noticing that commutative basic algebras share many properties with MV-algebras; especially, finite commutative basic algebras coincide with finite MV-algebras, thus the logic $\mathscr{L}_{CBA}$ and the Łukasiewicz logic have the same finite models (see [20]).

4.1.1 Commutative basic algebras

In what follows, we restrict ourselves to commutative basic algebras (unless we say otherwise). We have already pointed out in the introduction that commutative basic algebras are similar to MV-algebras; for instance, the underlying lattice of a commutative basic algebra is distributive (cf. [35]) and every finite commutative basic algebra is actually an MV-algebra (cf. [20]). Moreover, in commutative basic algebras on the real interval $[0, 1]$, the operation $\odot$ defined by $x \odot y = \neg(\neg x \oplus \neg y)$ is a semicopula [63] and (up to isomorphism) the negation is given by $\neg x = 1 - x$ (see [17]). An example of a proper commutative basic algebra which is not an MV-algebra can be found in [17].

We define a term operations $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \to y := \neg x \oplus y$. Those operation satisfy the adjointness property

$$x \odot y \leq z \text{ if and only if } x \leq y \to z.$$

[a] We should stress that our basic algebras and Hájek's BL-algebras are two independent notions; in fact, the intersection of basic algebras and BL-algebras are just MV-algebras.

It is useful to work with another term operation which we call subtraction and which is defined by

$$x \ominus y = \neg(\neg x \oplus y).$$

The underlying partial order is then given by $x \leq y$ iff $x \ominus y = 0$, and we have

$$x \vee y = (x \ominus y) \oplus y \quad \text{and} \quad x \wedge y = x \ominus (x \ominus y). \tag{4.1}$$

We now list some arithmetical properties of commutative basic algebras; the proofs are straightforward and thus left to the reader.

Lemma 4.1. *In any commutative basic algebra* $\mathbf{A}$*, for all* $x, y, z \in A$*:*

(i) *if* $x \leq y$*, then* $x \oplus z \leq y \oplus z$*,* $z \ominus y \leq z \ominus x$ *and* $x \ominus z \leq y \ominus z$*,*

(ii) $(x \oplus y) \ominus y = x \wedge \neg y$*,*

(iii) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$*,*

(iv) $x \oplus y = (x \wedge \neg y) \oplus y$*,*

(v) $(x \oplus y) \wedge z \leq (x \wedge z) \oplus (y \wedge z)$*,*

(vi) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$*,*

(vii) $(x \vee y) \ominus y = x \ominus y = x \ominus (x \wedge y)$*,*

(viii) $(x \wedge y) \oplus (x \ominus y) = x$*,*

(ix) $(x \ominus y) \wedge (y \ominus x) = 0$ *(prelinearity).*

We close this paragraph with an alternative characterization of MV-algebras in the setting of basic algebras:

Lemma 4.2. *A commutative basic algebra is an MV-algebra if and only if it satisfies the identity*

$$(z \ominus y) \ominus (z \ominus x) \leq x \ominus y. \tag{4.2}$$

Proof. Let A be a commutative basic algebra satisfying (4.2). Then for $x, y, z \in A$ we have $(\neg z \ominus y) \ominus x \geq (\neg z \ominus x) \ominus (\neg z \ominus (\neg z \ominus y)) = (\neg z \ominus x) \ominus (\neg z \wedge y) \geq (\neg z \ominus x) \ominus y$ by (4.1) and Lemma 4.1 (i). When interchanging x and y, we get the converse inequality, hence $\neg((z \oplus y) \oplus x) = (\neg z \ominus y) \ominus x = (\neg z \ominus x) \ominus y = \neg((z \oplus x) \oplus y)$. Thus $\mathbf{A}$ satisfies the identity $(z \oplus y) \oplus x = (z \oplus x) \oplus y$, and so $\oplus$ is associative.

The other direction is easy: if $\mathbf{A}$ is an MV-algebra, then $(z \ominus y) \ominus (z \ominus x) = \neg(\neg z \oplus y \oplus \neg(\neg z \oplus x)) = \neg(y \oplus (\neg x \vee \neg z)) \leq \neg(y \oplus \neg x) = x \ominus y$. □

4.1.2 Ideals in commutative basic algebras

A *preideal* of a basic algebra $\mathbf{A}$ is a non-empty subset I such that

(i) $a \oplus b \in I$ for all $a, b \in I$, and

(ii) for every $a \in I$ and $b \in A$, if $b \leq a$, then also $b \in I$.

An *ideal*[b] of A is then a subset which is the 0-class of some congruence on A (see [111]). We denote by $\mathcal{P}(\mathbf{A})$ and $\mathcal{I}(\mathbf{A})$ the set of preideals of $\mathbf{A}$ and the set of ideals of $\mathbf{A}$, respectively.

As proved in [35], the variety of basic algebras is congruence regular (see [31]), i.e., every congruence is determined by its arbitrary class, and hence an ideal I is the 0-class of a unique congruence on $\mathbf{A}$, say Θ_I, which is specified by I as follows:

$$(a, b) \in \Theta_I \quad \text{iff} \quad a \ominus b, b \ominus a \in I. \tag{4.3}$$

This means that $\mathcal{I}(\mathbf{A})$ ordered by set-inclusion forms a complete lattice that is isomorphic to the congruence lattice $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$ under the assignment $I \mapsto \Theta_I$, the inverse of which is $\Theta \mapsto [0]_\Theta$. Accordingly, given an ideal

[b] A terminological note: We reserve the term "ideal" for the 0-classes of congruences. In MV-algebras, ideals and preideals coincide, but in basic algebras, the conditions (i) and (ii) are not enough for I to be the 0-class of a congruence. Moreover, the term "MV-ideal" instead of "ideal" is sometimes used in the literature on MV-algebras. We would like to warn the reader that our MV-ideals defined below (see Lemma 4.6 and the definition before it) are something else.

$I \in \mathcal{I}(\mathbf{A})$, we simply write $\mathbf{A}/I$ for the quotient algebra $\mathbf{A}/\Theta_I$, and a/I for the congruence class a/Θ_I. It is obvious that in $\mathbf{A}/I$ we have $a/I \leq b/I$ iff $a \ominus b \in I$.

Likewise $\mathcal{P}(\mathbf{A})$ is a complete lattice under set-inclusion; by [111], it is a distributive lattice which contains $\mathcal{I}(\mathbf{A})$ as a complete sublattice. It is readily seen that the preideal generated by $\emptyset \neq B \subseteq A$ contains exactly those elements $a \in A$ such that $a \leq \tau(b_1, \ldots, b_n)$ for some n-ary additive term[c] τ and $b_1, \ldots, b_n \in B$.

Lemma 4.3. *Let* $\mathbf{A}$ *be a commutative basic algebra. Then* $I \subseteq A$ *is a preideal iff* $0 \in I$ *and, for all* $a, b \in A$, *if* $a \ominus b, b \in I$, *then* $a \in I$.

Proof. If I is a preideal, then $a \ominus b, b \in I$ implies $a \vee b = (a \ominus b) \oplus b \in I$, whence $a \in I$. Conversely, if $a, b \in I$, then $((a \oplus b) \ominus b) \ominus a = (a \wedge \neg b) \ominus a = 0 \in I$ which yields $a \oplus b \in I$. Also, I is downwards closed since if $a \leq b$ and $b \in I$, then $a \ominus b = 0 \in I$, which implies $a \in I$. □

We now give an internal characterization of ideals of commutative basic algebras (cf. [18], [39]).

Lemma 4.4. *In any commutative basic algebra* $\mathbf{A}$, *for every* $I \in \mathcal{P}(\mathbf{A})$, *the following are equivalent:*

(i) I *is an ideal of* $\mathbf{A}$;

(ii) $(a \oplus (b \oplus x)) \ominus (a \oplus b) \in I$ *for all* $a, b \in A$ *and* $x \in I$;

(iii) for all $a, b, c \in A$, *if* $a \ominus b \in I$, *then* $(c \oplus a) \ominus (c \oplus b) \in I$;

(iv) for all $a, b, c \in A$, *if* $a \ominus b \in I$, *then* $(c \ominus b) \ominus (c \ominus a) \in I$.

Proof. (i) $\Rightarrow$ (ii) Let I be an ideal, i.e. $I = [0]_\Theta$ where $\Theta \in \mathrm{Con}(A)$. If $x \in I$, then $(x, 0) \in \Theta$ implies $(a \oplus (b \oplus x)) \ominus (a \oplus b)\ \Theta\ (a \oplus (b \oplus 0)) \ominus (a \oplus b) = 0$, so $(a \oplus (b \oplus x)) \ominus (a \oplus b) \in I$.

(ii) $\Rightarrow$ (iii). If $a \ominus b \in I$, then by Lemma 4.1 (i) and (4.1)

$$(c \oplus a) \ominus (c \oplus b) \leq (c \oplus (a \vee b)) \ominus (c \oplus b) = [c \oplus (b \oplus (a \ominus b))] \ominus (c \oplus b) \in I,$$

[c] By an *additive term* we mean a term in which $\neg$ does not occur, i.e., τ is built from the variables (and the constant 0) using only the addition $\oplus$.

whence $(c \oplus a) \ominus (c \oplus b) \in I$.

(iii) $\Leftrightarrow$ (iv). It suffices to observe that $(c \ominus b) \ominus (c \ominus a) = \neg(\neg c \oplus b) \ominus \neg(\neg c \oplus a) = (\neg c \oplus a) \ominus (\neg c \oplus b)$ for every $c \in A$.

(iii)/(iv) $\Rightarrow$ (i). We prove that Θ_I defined by (4.3) is a congruence whose 0-class is I. We have $(a, 0) \in \Theta_I$ iff $a \in I$, i.e. $[0]_{\Theta_I} = I$, and Θ_I is obviously reflexive, symmetrical and compatible with $\neg$ and $\oplus$ by (iii). So there remains to show that it is transitive: If (x, y), $(y, z) \in \Theta_I$, then $y \ominus z \in I$ implies $(x \ominus z) \ominus (x \ominus y) \in I$ whence $x \ominus z \in I$ since $x \ominus y \in I$. Similarly, from $y \ominus x, z \ominus y \in I$ we get $(z \ominus x) \ominus (z \ominus y) \in I$, and consequently, $z \ominus x \in I$. Thus $(x, z) \in \Theta_I$. $\square$

Lemma 4.5. *Let* $\mathbf{A}$ *be a commutative basic algebra and* $I \in \mathcal{I}(\mathbf{A})$*. Then, for all* $a, b \in A$*,*

$$a \oplus (b \oplus I) = (a \oplus b) \oplus I.\text{[d]} \tag{4.4}$$

Proof. Since I is an ideal of $\mathbf{A}$, by Lemma 4.4 (ii) we have $(a \oplus (b \oplus x)) \ominus (a \oplus b) \in I$ for all $a, b \in A$ and $x \in I$, and hence

$$a \oplus (b \oplus x) = (a \oplus (b \oplus x)) \vee (a \oplus b) = [(a \oplus (b \oplus x)) \ominus (a \oplus b)] \oplus (a \oplus b) \in I \oplus (a \oplus b),$$

proving $a \oplus (b \oplus I) \subseteq I \oplus (a \oplus b) = (a \oplus b) \oplus I$. On the other hand, if $x \in I$, then

$$(((a \oplus b) \oplus x) \ominus a) \ominus b \;\Theta_I\; ((a \oplus b) \ominus a) \ominus b = (\neg a \wedge b) \ominus b = 0,$$

and so $(((a \oplus b) \oplus x) \ominus a) \ominus b \in I$. Then

$$\begin{aligned}
(a \oplus b) \oplus x &= ((a \oplus b) \oplus x) \vee a \vee (a \oplus b) \\
&= [(((a \oplus b) \oplus x) \ominus a) \oplus a] \vee (b \oplus a) \\
&= [(((a \oplus b) \oplus x) \ominus a) \vee b] \oplus a \\
&= ([(((a \oplus b) \oplus x) \ominus a) \ominus b] \oplus b) \oplus a \\
&\in (I \oplus b) \oplus a = a \oplus (b \oplus I),
\end{aligned}$$

which proves the converse inclusion. $\square$

[d] Here $a \oplus (b \oplus I) = \{a \oplus (b \oplus x) \mid x \in I\}$ and $(a \oplus b) \oplus I = \{(a \oplus b) \oplus x \mid x \in I\}$.

By the *polar* of a non-empty set X in a basic algebra $\mathbf{A}$ we mean the set

$$X^\perp = \{a \in A \mid a \wedge x = 0 \text{ for all } x \in X\}.$$

Writing $x^\perp$ instead of $\{x\}^\perp$ we have $X^\perp = \bigcap_{x\in X} x^\perp$; Lemma 4.1 (v) then entails that $X^\perp$ is a preideal[e] of $\mathbf{A}$. Moreover, for every preideal I of $\mathbf{A}$, the polar $I^\perp$ is the pseudocomplement of I in the lattice $\mathcal{P}(\mathbf{A})$ of preideals of $\mathbf{A}$ (see [111]). We write $X^{\perp\perp}$ for $(X^\perp)^\perp$.

It is worth observing that $x^{\perp\perp} \cap y^{\perp\perp} = \{0\}$ whenever $x \wedge y = 0$. Indeed, if $x \wedge y = 0$, then $x \in y^\perp$ which implies $y^{\perp\perp} \subseteq x^\perp$, whence $x^{\perp\perp} \cap y^{\perp\perp} \subseteq x^{\perp\perp} \cap x^\perp = \{0\}$.

Theorem 4.1. *Let* $\mathbf{A}$ *be a commutative basic algebra. Then* $\mathbf{A}$ *is* representable *(i.e.,* $\mathbf{A}$ *is a subdirect product of linearly ordered commutative basic algebras) if and only if* $\mathbf{A}$ *satisfies the identity*

$$[(x \oplus (y \oplus (z \ominus u))) \ominus (x \oplus y)] \wedge (u \ominus z) = 0. \tag{4.5}$$

Proof. The satisfaction of (4.5) amounts to saying that every polar in $\mathbf{A}$ is an ideal in $\mathbf{A}$. Indeed, if polars are ideals, then $z \ominus u \in (u \ominus z)^\perp$ (this is just a reformulation of prelinearity) implies $(x \oplus (y \oplus (z \ominus u))) \ominus (x \oplus y) \in (u \ominus z)^\perp$. Conversely, if $\mathbf{A}$ fulfills (4.5), then $z \in u^\perp$ yields $z \ominus u = z$ and $u \ominus z = u$, and consequently, $(x \oplus (y \oplus z)) \ominus (x \oplus y) \in u^\perp$. Thus $u^\perp$ is an ideal by Lemma 4.4 (ii).

Any representable basic algebra satisfies (4.5) since the identity is satisfied by all linearly ordered basic algebras where we either have $z \ominus u = 0$ or $u \ominus z = 0$. For the converse, assume that A satisfies (4.5). By Birkhoff's theorem (see e.g. [27], Theorem 8.6), A is a subdirect product of subdirectly irreducible algebras which are homomorphic images of $\mathbf{A}$, i.e. commutative basic algebras satisfying (4.5). Hence it suffices to show that if $\mathbf{A}$ is subdirectly irreducible, then it is linearly ordered. If $\mathbf{A}$ is not linearly ordered, then there exist $a, b \in A$ such that $a \ominus b \neq 0 \neq b \ominus a$ and $(a \ominus b) \wedge (b \ominus a) = 0$. Then $(a \ominus b)^{\perp\perp}$ and $(b \ominus a)^{\perp\perp}$ are non-zero ideals with

[e] We do not claim that $X^\perp$ is an ideal, it is a preideal! The situation when all polars are ideals is described in Theorem 4.1.

$(a \ominus b)^{\perp\perp} \cap (b \ominus a)^{\perp\perp} = \{0\}$ (by the observation before the theorem), and since the ideal lattice $\mathcal{I}(\mathbf{A})$ is isomorphic to the congruence lattice $\mathrm{Con}(\mathbf{A})$, it follows that $\mathbf{A}$ cannot be subdirectly irreducible. □

Given a basic algebra $\mathbf{A}$, we call an ideal I of $\mathbf{A}$ an *MV-ideal*[f] provided that the quotient algebra $\mathbf{A}/I$ is an MV-algebra. Since MV-algebras are characterized as basic algebras satisfying the identity (4.2), it is plain that $I \in \mathcal{I}(\mathbf{A})$ is an MV-ideal iff

$$[(c \ominus b) \ominus (c \ominus a)] \ominus (a \ominus b) \in I$$

for all $a, b, c \in A$. It follows that the set $\mathcal{MV}(A)$ of all MV-ideals of $\mathbf{A}$ is a complete lattice under set-inclusion. The MV-ideal generated by $\emptyset \neq X \subseteq A$ is denoted by $MV(X)$ and can be described as follows:

Lemma 4.6. *In a commutative basic algebra $\mathbf{A}$, a preideal $I \in \mathcal{P}(\mathbf{A})$ is an MV-ideal iff $[(c \ominus b) \ominus (c \ominus a)] \ominus (a \ominus b) \in I$ for all $a, b, c \in A$. Moreover, for every $\emptyset \neq X \subseteq A$, $MV(X)$ is the preideal generated by the set $X \cup \{[(c \ominus b) \ominus (c \ominus a)] \ominus (a \ominus b) \mid a, b, c \in A\}$.*

Proof. Let I be a preideal such that $[(c \ominus b) \ominus (c \ominus a)] \ominus (a \ominus b) \in I$ for all $a, b, c \in A$. Then $(c \ominus b) \ominus (c \ominus a) \in I$ whenever $a \ominus b \in I$. Thus I fulfills the condition (iv) of Lemma 4.4, and hence it is an ideal of $\mathbf{A}$.

The statement about $MV(X)$ is straightforward. □

For every $a \in A$ we define $N(a)$ to be the smallest subset of $\mathbf{A}$ such that $a \in N(a)$ and $x \oplus y \in N(a)$ for all $x, y \in N(a)$. In other words, the elements of $N(a)$ are of the form $\tau(A; \ldots, a)$ where τ is an additive term; for example, $a \oplus a$, $(a \oplus a) \oplus a$, $a \oplus (a \oplus a)$, $(a \oplus a) \oplus (a \oplus a)$ etc. belong to $N(a)$.

Lemma 4.7. *Let $\mathbf{A}$ be a commutative basic algebra. For every $I \in \mathcal{MV}(A)$ and $x \in A$,*

$$MV(I \cup \{x\}) = \{a \in A \mid a \leq b \oplus c \text{ where } b \in I \text{ and } c \in N(x)\}.$$

[f] Clearly, in MV-algebras we have MV-ideals = ideals = preideals, while in case of basic algebras we have generally strict inclusions MV-ideals $\subset$ ideals $\subset$ preideals.

Proof. Since I is an MV-ideal, it contains all the elements $[(c \ominus b) \ominus (c \ominus a)] \ominus (a \ominus b)$ for $a, b, c \in A$, and hence, by Lemma 4.6, $a \in MV(I \cup \{x\})$ iff $a \leq y$ where y is a sum of elements of $I \cup \{x\}$. However, using the property (4.4), we can rewrite y as $b \oplus c$ where $b \in I$ and $c \in N(x)$. □

We say that an ideal $I \in \mathcal{I}(\mathbf{A})$ is *proper* if $I \neq A$. If there is no proper ideal strictly exceeding I, then I is *maximal.* MV-ideals which are maximal ideals can be characterized in a similar way in which maximal ideals of MV-algebras are described:

Corollary 4.1. *Let* $\mathbf{A}$ *be a commutative basic algebra and* I *its proper MV-ideal. The following are equivalent:*

(i) I *is a maximal ideal;*

(ii) for every $x \in A$, $x \notin I$ *iff* $\neg y \in I$ *for some* $y \in N(x)$.

Proof. (i) $\Rightarrow$ (ii). First, if $x \notin I$, then $MV(I \cup \{x\}) = A$, so $1 = b \oplus c$ for some $b \in I$ and $c \in N(x)$. Hence $\neg c \leq b$, which yields $\neg c \in I$. Second, let $x \in I$ and suppose that $\neg y \in I$ for some $y \in N(x)$. Then $1 = y \oplus \neg y \in I$ since $N(x) \subseteq I$. But this contradicts the assumption $I \neq A$.

(ii) $\Rightarrow$ (i). Let J be an ideal with $I \subset J$. For every $x \in J \setminus I$ there exists $y \in N(x)$ such that $\neg y \in I$. Since $N(x) \subseteq J$, we have $1 = \neg y \oplus y \in J$, which entails $J = I$. Thus I is a maximal ideal. □

4.2 Examples of non-associative commutative basic algebras

It has been show in [20] that every finite commutative basic algebra is an MV-algebra. On the other hand, the variety of commutative basic algebras does not coincide with the variety of MV-algebras as it was shown by in [17]. In fact, it will be shown that there exists a commutative basic algebra on interval $[0, 1]$ of reals which is subdirectly irreducible and is not an MV-algebra. Of course, it is linearly ordered.

Hence, it is a natural question for which cardinalities there exist totally-ordered basic algebras which are not MV-algebras. The problem of existence of totally ordered MV-algebras of an arbitrary cardinality was solved recently by P. Wojciechowski [145]. It motivated us to find a similar construction. P. Wojciechowski used a famous Hahn Embedding Theorem saying that every abelian o-group embeds into the Hahn group on some totally ordered set. Since every MV-algebra can be constructed as an interval of a lattice ordered abelian group (see e.g. [46]), this machinery works also for MV-algebras. On the contrary, commutative basic algebras are not related to ordered groups. Hence, our approach is rather different. We use the construction by P. Wojciechowski to obtain infinite MV-algebra and we use another algebraic construction to change this MV-algebra into a commutative basic algebra which is not an MV-algebra but which is still totally ordered and of the same cardinality. Then, applying recent results from [21], [20], it can be shown easily that the resulting basic algebra is, moreover, subdirectly irreducible.

Among other things this shows that the variety of (commutative) basic algebras is not residually small. On the contrary, it has a proper class of SI-members of an arbitrary infinite cardinality.

In any basic algebra $\mathbf{A} = (A; \oplus, \neg, 0)$, one can consider the term operation $x \to y := \neg x \oplus y$. Of course, we can derive also conversely $x \oplus y = \neg x \to y$ due to the double negation law (BA2). Hence, $\mathbf{A}$ can be described alternatively in the operations $\{\to, 0\}$ where $\neg x = x \to 0$. Recall that by *chain basic algebra* is meant a basic algebra which is a chain with respect to the induced partial order. The following assertion is easy to check.

Proposition 4.1. *(1) Let* $\mathbf{A} = (A; \oplus, \neg, 0)$ *be a chain basic algebra. Then*

(i) $x \leq y$ *implies* $x \to y = 1$,

(ii) $x \leq y$ *implies* $(y \to x) \to x = y$,

(iii) if $y \leq x \leq z$ *then* $z \to y \leq x \to y$.

If $\mathbf{A}$ *is, moreover, commutative then* *(iv)* $x \to y = (y \to 0) \to (x \to 0) = \neg y \to \neg x.$

(2) Let $(A; \leq)$ *be an ordered set with a least element* 0 *and a greast element* 1*. Let* $\to$ *be a binary operation on* $\mathbf{A}$ *satisfying (i), (ii) and (iii). Define* $\neg x = x \to 0$ *and* $x \oplus y = \neg x \to y$*. Then* $\mathbf{A} = (A; \oplus, \neg, 0)$ *is a chain basic algebra. If, moreover,* $\to$ *satisfies (iv) then* $\mathbf{A}$ *is commutative.*

It is evident that a commutative basic algebra is an MV-algebra if and only if the operation $\oplus$ is associative. Using the operation $\to$, it can be expressed as follows: A basic algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ is an MV-algebra if and only if satisfies the Exchange Identity

$$x \to (y \to z) = y \to (x \to z) \tag{E}$$

(se e.g. [34]).

We denote by $\mathbb{R}$ or $\mathbb{Q}$ the set of all real or rational numbers. By $+$ or $-$ we mean the arithmetical operations on reals. By $\mathbb{R}^+$ or $\mathbb{Q}^+$ is meant the interval $[0, \infty) \cap \mathbb{R}$ or $[0, \infty) \cap \mathbb{Q}$. Analogously, by $\mathbb{R}^-$ or $\mathbb{Q}^-$ is meant the interval $(\infty, 0] \cap \mathbb{R}$ or $(\infty, 0] \cap \mathbb{Q}$.

Let $\mathbf{A}$ be a linearly ordered MV-algebra. Denote by

$$A(\mathbb{R}) = (\{1\} \times \mathbb{R}^-) \cup (\{0\} \times \mathbb{R}^+) \cup (\{A \setminus \{0,1\}\} \times \mathbb{R}),$$
$$A(\mathbb{Q}) = (\{1\} \times \mathbb{Q}^-) \cup (\{0\} \times \mathbb{Q}^+) \cup (\{A \setminus \{0,1\}\} \times \mathbb{Q}).$$

Let us introduce the order $\leq$ on the sets $A(\mathbb{R})$ and $A(\mathbb{Q})$ such that $(a, x) \leq (b, y)$ if and only if $a \leq b$ (in $\mathbf{A}$) or $a = b$ and $x \leq y$ (as numbers). We introduce the operations $\oplus$ and $\neg$ on $\mathbf{A}(\mathbb{R})$ or $\mathbf{A}(\mathbb{Q})$ as follows:

$$\neg(x, y) := (\neg x, -y),$$

$$(x, y) \oplus (z, v) := \begin{cases} (x \oplus z, y + v) & \text{if} \quad \begin{array}{c} x \oplus z \neq 1 \text{ or} \\ \neg x = z \text{ and } \ y + v \leq 0 \end{array} \\ \\ (1, 0) & \text{otherwise.} \end{cases}$$

It is easy to check that $A(\mathbb{R})$ and $A(\mathbb{Q})$ are linearly ordered MV-algebras with the least element $(0,0)$ and the greatest element $(1,0)$ with respect to the lexicographic order on $A(\mathbb{R})$ and $A(\mathbb{Q})$.

Moreover, if A is of an infinite cardinality (or of the cardinality at least continuum) then $A(\mathbb{Q})$ (or $A(\mathbb{R})$, respectively) is of the same cardinality.

If $x \to y = \neg x \oplus y$ is derived in the standard MV-algebra on the interval $[0,1]$ of reals., then:

$$x \to y := \begin{cases} 1 - x + y & \text{if} \quad x \leq y \\ \\ 1 & \text{otherwise.} \end{cases}$$

To obtain a commutative basic algebra which is not an MV-algebra, the operation $\to$ must be changed. We are able to do it by means of a new function of deformation $d(x,y)$ which is described in [17]. The operation $\to$ is defined now by

$$x \to y := \begin{cases} 1 - x + y + d(x,y) & \text{if} \quad x \leq y \\ \\ 1 & \text{otherwise.} \end{cases}$$

However, $d(x,y)$ must be picked up in the way to satisfy axioms of commutative basic algebra. If $d(x,y) = 0$ for all reals x, y then resulting algebra is the standard MV-algebra.

We generalize this idea for construction of system of deformation functions on MV-algebras $A(\mathbb{R})$ or $A(\mathbb{Q})$ such that new algebra is commutative basic algebra, but it is not an MV-algebra. The operation $\to$ on $A(\mathbb{R})$ and $A(\mathbb{Q})$ can be defined as

$$(a,x) \to (b,y) := \begin{cases} (a \to b, y - x) & \text{if} \quad (a,x) \geq (b,y) \\ \\ (1,0) & \text{otherwise.} \end{cases}$$

To find the system of deformation functions we must firstly prove the following theorem.

Theorem 4.2. *Let* $\mathbf{A}$ *be an MV-algebra and* $a \to b := \neg a \oplus b$. *On* $A(\mathbb{R})$ *or* $A(\mathbb{Q})$ *we define*

$$(a,x) \to (b,y) := \begin{cases} (a \to b, y - x + d_{a,b}(x,y)) & \textit{if} \quad (a,x) \geq (b,y) \\ \\ (1,0) & \textit{otherwise,} \end{cases}$$

where $\{d_{a,b} \mid a, b \in A; b \leq a\}$ *is a system of functions* $d_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *or* $d_{a,b} : \mathbb{Q}^2 \longrightarrow \mathbb{Q}$ *which satisfy for* $(b,y) \leq (a,x) < (a,z)$ *the following conditions*

(Di) $d_{a,0}(x,0) = 0,$

(Dii) $d_{a,b}(x,y) = d_{a\to b,b}(y - x + d_{a,b}(x,y), y),$

(Diii) $d_{a,b}(x,y) = d_{\neg b,\neg a}(-y,-x),$

(Div) $\frac{d_{a,b}(z,y) - d_{a,b}(x,y)}{z-x} < 1.$

Define $(a,x) \oplus (b,y) := ((a,x) \to (0,0)) \to (b,y)$ *and* $\neg(a,x) = (a,x) \to (0,0)$. *Then the algebras*

$$\mathbf{A}(\mathbb{R}) = (A(\mathbb{R}), \oplus, \neg, (0,0))$$

and

$$\mathbf{A}(\mathbb{Q}) = (A(\mathbb{Q}), \oplus, \neg, (0,0))$$

are linearly ordered commutative basic algebras.

Proof. By Proposition 4.1 we have to check the conditions (i)-(iv). Firstly, we can easily check that $\neg(a,x) = (a,x) \to (0,0) = (a \to 0, 0 - x + d_{a,0}(x,0)) = (\neg a, -x)$.
(i) Clearly, $(a,x) \to (b,y) = (1,0) = \neg(0,0)$, if $(a,x) \geq (b,y)$ holds.
(ii) Let $(a,x) \geq (b,y)$. Then we can write

$$\begin{aligned}((a,x)\to(b,y))\to(b,y) &= (a\to b, y-x+d_{a,b}(x,y))\to(b,y) = \\ &((a\to b)\to b, y-(y-x+d_{a,b}(x,y))+ \\ &d_{a\to b,b}(y-x+d_{a,b}(x,y),y)) = \\ &(a\vee b, x-d_{a,b}(x,y))+ \\ &d_{a\to b,b}(y-x+d_{a,b}(x,y),y)) = \\ &(a,x)\end{aligned}$$

(iii) Consider the elements $(b,y)\leq(a,x)<(c,z)$. If $a<c$ then clearly $c\to b<a\to b$ and also $(c,z)\to(b,y)=(c\to b, y-z+d_{c,b}(z,y))<(a\to b, y-x+d_{a,b}(x,y))=(a,x)\to(b,y)$.

If $a=c$ then the inequality $(b,y)\leq(a,x)<(a,z)$ is satisfied and $z-x>0$. Then

$$\frac{d_{a,b}(z,y)-d_{a,b}(x,y)}{z-x}<1$$

yields $d_{a,b}(z,y)-z+y<d_{a,b}(x,y)-x+y$ and finally $(a,z)\to(b,y)=(a\to b, y-z+d_{a,b}(z,y))<(a\to b, y-x+d_{a,b}(x,y))=(a,x)\to(b,y)$.

(iv) Let $(a,x)\geq(b,y)$. Then we have

$$\begin{aligned}\neg(b,y)\to\neg(a,x) &= (\neg b,-y))\to(\neg a,-x) = \\ &(\neg b\to\neg a, -y-(-x)+d_{\neg b,\neg a}(-y,-x)) = \\ &(a\to b, x-y+d_{a,b}(x,y)) = \\ &(a,x)\to(b,y).\end{aligned}$$

□

The following facts taken from [21],[20],[35],[33] will be employed in the next proof.

Fact 1. If $\mathbf{A}=(A;\oplus,\neg,0)$ is a basic algebra and $\theta\in$ *Con* $\mathbf{A}$ then $x,y\in 0/\theta$ yields $x\oplus y\in 0/\theta$.

Fact 2. If $\bigvee\{x_i \mid i \in I\}$ exists in commutative basic algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ then there exists also $\bigvee\{y \oplus x_i \mid i \in I\}$ for any $y \in A$ and

$$y \oplus \bigvee\{x_i \mid i \in I\} = \bigvee\{y \oplus x_i \mid i \in I\}.$$

Fact 3. Since $\oplus$-idempotent elements in any basic algebra are just the boolean elements, any chain basic algebra has only two such elements which are 0 and 1.

Theorem 4.3. *Commutative basic algebras* $\mathbf{A}(\mathbb{R})$ *and* $\mathbf{A}(\mathbb{Q})$ *constructed in Theorem 4.2 are subdirectly irreducible.*

Proof. As pointed in [34], every congruence on basic algebra is fully determined by its ideal. Hence, to prove that a basic algebra $\mathbf{A}$ is subdirectly irreducible, it is enough to show that it contains a nontrivial least ideal. The set $I = \{(0, x) \mid x \in \mathbb{R}^+\}$ is the kernel of the projection $h(a, x) = a$ (which is a homomorphismus) and hence $I = (0, 0)/\theta_h$ for the induced congruence θ_h. Thus I is an ideal of $\mathbf{A}$ (in the sense of [21]). It remains to show that I is the least non-singleton ideal. Assume J is an ideal of $\mathbf{A}(\mathbb{R})$ or $\mathbf{A}(\mathbb{Q})$ such that $J \subset I$. If J is non-trivial then there exists $x \in J$ such that $x \neq 0$. Existence of suprema in real numbers shows that $\bigvee J$ exists in $\mathbf{A}(\mathbb{R})$. However, $\mathbf{A}(\mathbb{Q})$ is subalgebra of $\mathbf{A}(\mathbb{R})$ and we can compute in $\mathbf{A}(\mathbb{R})$:

$$\bigvee J \leq \bigvee J \oplus \bigvee J = \bigvee_{x \in J} (x \oplus \bigvee J) = \bigvee_{x \in J} \bigvee_{y \in J} (x \oplus y) \leq \bigvee J$$

thus $\bigvee J$ is an idempotent element of $\mathbf{A}(\mathbb{R})$. Of course, $\bigvee J \neq (1, 0)$ thus we have $\bigvee J = (0, 0)$ proving that J is a singleton. Hence, I is a non-trivial least ideal and thus $\mathbf{A}(\mathbb{R})$ is subdirectly irreducible. Although $\bigvee J$ need not belong to $\mathbf{A}(\mathbb{Q})$, the ideal I is the least non trivial ideal of $\mathbf{A}(\mathbb{Q})$ and hence both $\mathbf{A}(\mathbb{Q})$ and $\mathbf{A}(\mathbb{R})$ are subdirectly irreducible. $\square$

Theorem 4.4. *For every MV-algebra* $\mathbf{A}$*, there exists a system of nonzero functions* $\{d_{a,b} \mid a, b \in A\}$ *on* $\mathbb{R}$ *or* $\mathbb{Q}$ *satisfying the conditions (Di)-(Div) of Theorem 4.4.*

Proof. We can define
$$d_{a,b}(x,y) = 0$$
for all $(a,b) \notin \{(0,0),(1,0),(1,0)\}$. Hence, for such (a,b) the operation $(a,x) \to (b,y))$ satisfies trivially the conditions (i)-(iv) of Proposition 4.1. For $(a,b) \in \{(0,0),(1,0),(1,0)\}$ we define $d_{a,b}$ similary as it was done by the first author in [17], i.e.:

$d_{1,1} : (\mathbb{R}^-)^2 \longrightarrow \mathbb{R}$ (resp. $d_{1,1} : (\mathbb{Q}^-)^2 \longrightarrow \mathbb{Q}$).

$$d_{1,1}(x,y) := \begin{cases} \frac{1}{6}x - \frac{1}{6}y & \text{if} \quad x \geq y > \frac{3}{2}x \\ -\frac{1}{3}x + \frac{1}{6}y & \text{if} \quad \frac{3}{2}x \geq y > 2x \\ \frac{1}{4}x - \frac{1}{8}y & \text{if} \quad 2x \geq y > \frac{18}{5}x \\ -\frac{1}{5}x & \text{if} \quad \frac{18}{5}x \geq y \end{cases}$$

$d_{0,0} : (\mathbb{R}^+)^2 \longrightarrow \mathbb{R}$ (resp. $d_{0,0} : (\mathbb{Q}^+)^2 \longrightarrow \mathbb{Q}$).

$$d_{0,0}(x,y) := \begin{cases} \frac{1}{6}x - \frac{1}{6}y & \text{if} \quad x \geq y > \frac{2}{3}x \\ -\frac{1}{6}x + \frac{1}{3}y & \text{if} \quad \frac{2}{3}x \geq y > \frac{1}{2}x \\ \frac{1}{8}x - \frac{1}{4}y & \text{if} \quad \frac{1}{2}x \geq y > \frac{5}{18}x \\ \frac{1}{5}y & \text{if} \quad \frac{5}{18}x \geq y \end{cases}$$

$d_{1,0} : \mathbb{R}^- \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ (resp. $d_{1,0} : \mathbb{Q}^- \times \mathbb{Q}^+ \longrightarrow \mathbb{Q}$).

$$d_{1,0}(x,y) := \begin{cases} -\frac{1}{5}x & \text{if} \quad y \geq -\frac{12}{5}x \\ -\frac{1}{7}x + \frac{1}{7}y & \text{if} \quad -\frac{12}{5}x \geq y > -x \\ \frac{1}{7}x - \frac{1}{7}y & \text{if} \quad -x \geq y > -\frac{5}{12}x \\ \frac{1}{5}y & \text{if} \quad -\frac{5}{12}x \geq y \end{cases}$$

It is elemetary to check that $d_{0,0}(x,y) = d_{1,1}(-y,-x)$ and $d_{1,0}(x,y) = d_{1,0}(-y,-x)$ (for example let $(x,y) \in (\mathbb{R}^-)^2$ such that $x \geq y > \frac{3}{2}x$, then clearly $(-y,-x) \in (\mathbb{R}^+)^2$ and also $-y \leq x \leq \frac{2}{3}(-y)$ thus $d_{1,1}(x,y) = \frac{1}{6}x - \frac{1}{6}y = \frac{1}{6}(-y) - \frac{1}{6}(-x) = d_{0,0}(-y,-x))$.

Conditions (i) and (iv) from Proposition 4.1 can be also checked by an elementary computation.

Analogously we can check that $d_{0,0}(x,y) = d_{1,0}(y - x + d_{0,0}(x,y), y)$, $d_{1,1}(x,y) = d_{1,1}(y-x+d_{1,1}(x,y), y)$ and $d_{1,0}(x,y) = d_{0,0}(y-x+d_{1,0}(x,y), y)$.

For example let $(x, y) \in (\mathbb{R}^-)^2$ such that $x \geq y > \frac{3}{2}x$. Then clearly $y - x + d_{1,1}(x, y) = \frac{5}{6}y - \frac{5}{6}x$ and $(\frac{5}{6}y - \frac{5}{6}x, y) \in (\mathbb{R}^-)^2$. The inequality $y > \frac{3}{2}x$ yields $\frac{18}{5}(\frac{5}{6}y - \frac{5}{6}x) > y$. Finally we compute

$$d_{1,1}(x, y) = \frac{1}{6}x - \frac{1}{6}y = -\frac{1}{5}(\frac{5}{6}y - \frac{5}{6}x) = d_{1,1}(y - x + d_{1,1}(x, y), y).$$

$\square$

Corollary 4.2. *For an arbirary infinite cardinal κ there exists a subdirectly irreducible linearly ordered commutative basic algebra of cardinality κ which is not an MV-algebra.*

Proof. Of course, if κ is an infinite cardinal then, by [145], there exists an MV-algebra $\mathbf{A}$ which is linearly ordered and of cardinality κ. Hence, $\mathbf{A}(\mathbb{Q})$ is also of cardinality κ. By Theorems 4.3 and 4.4, $\mathbf{A}(\mathbb{Q})$ is subdirectly irreducible and linearly ordered.

To prove that $\mathbf{A}(\mathbb{Q})$ is not an MV-algebra, we can show that the Exchange Identity (E) is violated. For the elements $(1, -4), (1, -5), (1, -10) \in \mathbf{A}(\mathbb{Q})$. We compute :

$$\begin{aligned}
(1, -4) \to \quad & ((1, -5) \to (1, -10)) = \\
& (1, -4) \to (1 \to 1, -10 + 5 + d_{1,1}(-5, -10)) = \\
& (1, -4) \to (1, -10 + 5 + \frac{1}{4}(-5) - \frac{1}{8}(-10)) = \\
& (1, -4) \to (1, -5) = \\
& (1 \to 1, -5 + 4 + d_{1,1}(-4, -5)) = \\
& (1, -5 + 4 + \frac{1}{6}(-4) - \frac{1}{6}(-5)) = \\
& (1, -\frac{5}{6}).
\end{aligned}$$

On the other side:

$$
\begin{aligned}
(1,-5) \to \quad & ((1,-4) \to (1,-10)) = \\
& (1,-5) \to (1 \to 1, -10+4+d_{1,1}(-4,-10)) = \\
& (1,-5) \to (1, -10+4+\frac{1}{4}(-4) - \frac{1}{8}(-10)) = \\
& (1,-5) \to (1, -\frac{23}{4}) = \\
& (1 \to 1, -\frac{23}{4}+5+d_{1,1}(-5,-\frac{23}{4})) = \\
& (1, -\frac{23}{4}+5+\frac{1}{6}(-4) - \frac{1}{6}(-\frac{23}{4})) = \\
& (1, -\frac{5}{8}).
\end{aligned}
$$

Thus, the exchange identity is not satisfied in algebra $\mathbf{A}(\mathbb{Q})$ and hence $\mathbf{A}(\mathbb{Q})$ is not an MV-algebra. $\square$

4.3 States on commutative basic algebras

States on MV-algebras were introduced by Mundici [120] as finitely additive measures of truth of propositions in the Łukasiewicz logic and play an important role in MV-algebraic probability theory (see [135]). Though the definition only requires finite additivity, i.e., $s(x \oplus y) = s(x) + s(y)$ whenever $x \leq \neg y$, there is a correspondence between states on an MV-algebra $\mathbf{A}$ and σ-additive regular Borel probability measures on the space $\mathcal{H}(\mathbf{A}) \subseteq [0,1]^A$ of the homomorphisms from A to the standard MV-algebra $[0,1]_{MV}$ (cf. [113], [112], [126]). The states on A form a convex compact subset of the Tychonoff cube $[0,1]^A$ the extreme boundary of which is just $\mathcal{H}(\mathbf{A})$. Moreover, the states on $\mathbf{A}$ coincide with those $[0,1]$-valued functions on A which satisfy de Finetti's coherence criterion with respect to $\mathcal{H}(\mathbf{A})$ (cf. [121], [113]).

Recently, the theory of states has been extended to algebras that generalize MV-algebras in the context of residuated lattices such as pseudo-MV-algebras by Dvurečenskij [64], BL-algebras by Riečan [134] and pseudo-BL-algebras by Georgescu [87], bounded divisible and semi-divisible residuated lattices by Dvurečenskij and Rachůnek [74], [75] and by Mertanen and Turunen [118], and bounded pseudo-BCK-algebras, which are the residuation subreducts of bounded residuated lattices, by Ciungu and Dvurečenskij [50]. The definition of a state for pseudo-MV-algebras is essentially the one given by Mundici, while for the other structures the definition of a Bosbach state by Georgescu [87] is used instead (in case of pseudo-MV-algebras, the two concepts coincide). From the logical point of view, in all these cases we may think of states as measures of "average degree of truth of a proposition" (cf. [120]) in a propositional logic with associative conjunction, i.e., a logic in which the formulas $(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$ and $\varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$ are provably equivalent.

Summarizing, we adopt Mundici's definition [120] and study states in the setting of commutative basic algebras that are the equivalent algebraic semantics for the non-associative fuzzy logic $\mathscr{L}_{CBA}$.

4.3.1 States

Following [120], by a *state* on a commutative basic algebra $\mathbf{A}$ we mean a $[0,1]$-valued function which is normalized and finitely additive; more precisely, a state on $\mathbf{A}$ is a function $s\colon A \to [0,1]$ such that

(i) $s(1) = 1$ (*normality*),

(ii) $s(a \oplus b) = s(a) + s(b)$ for all $a, b \in A$ with $a \leq \neg b$ (*additivity*).

The *kernel* of a state $s\colon A \to [0,1]$ is the set

$$Ker(s) = \{a \in A \mid s(a) = 0\},$$

and s is said to be a *faithful state* if $Ker(s) = \{0\}$.

In [120], additivity is phrased in a slightly different way: $s(a \oplus b) = s(a) + s(b)$ if $a \odot b = 0$. But $a \odot b = 0$ iff $\neg a \oplus \neg b = 1$ iff $a \leq \neg b$. The same holds true for basic algebras, but we avoid using $\odot$ in defining states because $\odot$ is not employed otherwise throughout the section.

Example 4.1. *Let $\mathbf{A}$ be an arbitrary commutative basic algebra. Then the direct product $\mathbf{A} \times [0,1]_{MV}$ is a commutative basic algebra which is an MV-algebra iff $\mathbf{A}$ is an MV-algebra. It is easily seen that the map $s\colon (a,x) \mapsto x$ is a state on $\mathbf{A} \times [0,1]_{MV}$ and $Ker(s) = \mathbf{A} \times \{0\}$. Clearly, we could have taken any subalgebra of $[0,1]_{MV}$ instead of $[0,1]_{MV}$.*

Example 4.2. *Let $\mathbf{A}$ be a commutative basic algebra and I an MV-ideal in $\mathbf{A}$. It is well-known that for every MV-algebra there exists a homomorphism to the standard MV-algebra $[0,1]_{MV}$; such a homomorphism is a state and hence every MV-algebra admits at least one state (cf. [46], [120]). Thus if m is a state on the MV-algebra $\mathbf{A}/I$, then $s\colon x \mapsto m(x/I)$ is a state on $\mathbf{A}$. We have $I \subseteq Ker(s)$, and $I = Ker(s)$ iff m is a faithful state.*

Lemma 4.8. *Let $\mathbf{A}$ be a commutative basic algebra and s a state on it. Then for all $a, b, c \in A$:*

(i) *$s(0) = 0$,*

(ii) *if $a \geq b$, then $s(a \ominus b) = s(a) - s(b)$ and $s(a) \geq s(b)$,*

(iii) $s(a \ominus b) = s(a) - s(a \wedge b) = s(a \vee b) - s(b)$,

(iv) $s(a \oplus b) \leq s(a) + s(b)$,

(v) $s((c \ominus b) \ominus (c \ominus a)) \leq s(a \ominus b)$.

Moreover, s is a faithful state if and only if, for all $a, b \in A$,

(vi) $a > b$ *implies* $s(a) > s(b)$.

Proof. (i) We have $1 = s(1) = s(1 \oplus 0) = s(1) + s(0) = 1 + s(0)$, so $s(0) = 0$.

(ii) If $a \geq b$, then $a = a \vee b = (a \ominus b) \oplus b$ where $a \ominus b = \neg b \ominus \neg a \leq \neg b$. Hence $s(a) = s(a \ominus b) + s(b)$. Now, since $0 \leq s(a \ominus b) = s(a) - s(b)$, we get $s(b) \leq s(a)$.

(iii) This follows directly from (ii) since $a \ominus b = a \ominus (a \wedge b) = (a \vee b) \ominus b$.

(iv) We have $s(a \oplus b) = s((a \wedge \neg b) \oplus b) = s(a \wedge \neg b) + s(b) \leq s(a) + s(b)$.

(v) Since $(a \vee c) \ominus (a \wedge b) \geq c \ominus (a \wedge b) \geq c \ominus b, c \ominus a$ by Lemma 4.1 (i), it follows $(c \ominus b) \ominus (c \ominus a) \leq [(a \vee c) \ominus (a \wedge b)] \ominus (c \ominus a)$ and by (ii) and (iii) we get $s((c \ominus b) \ominus (c \ominus a)) \leq s([(a \vee c) \ominus (a \wedge b)] \ominus (c \ominus a)) = s((a \vee c) \ominus (a \wedge b)) - s(c \ominus a) = s(a \vee c) - s(a \wedge b) - (s(a \vee c) - s(a)) = s(a) - s(a \wedge b) = s(a \ominus b)$.

(vi) Let s be faithful and suppose that $a > b$, but $s(a) \not> s(b)$. Then by (ii) we get $s(a) = s(b)$ and $s(a \ominus b) = s(a) - s(b) = 0$, whence $a \ominus b = 0$, so $a \leq b$, a contradiction. On the other hand, if s satisfies (vi), then it is faithful as $a > 0$ implies $s(a) > s(0) = 0$. $\square$

Proposition 4.2. *Let $\mathbf{A}$ be a commutative basic algebra. A function $s\colon A \to [0, 1]$ is a state on $\mathbf{A}$ if and only if $s(1) = 1$ and $s(a \ominus b) = s(a) - s(b)$ for all $a, b \in A$ with $a \geq b$.*

Proof. Let s be a $[0, 1]$-valued function satisfying the stated conditions. If $a \leq \neg b$, then $s(a \oplus b) = s(\neg(\neg b \ominus a)) = 1 - (1 - s(b) - s(a)) = s(a) + s(b)$. Thus s is a state. For the converse see Lemma 4.8 (ii). $\square$

Proposition 4.3. *Let $\mathbf{A}$ be a commutative basic algebra. The kernel $K = Ker(s)$ of every state s on $\mathbf{A}$ is a proper ideal of $\mathbf{A}$. In the quotient algebra $\mathbf{A}/K$ we have $a/K \leq b/K$ iff $s(a) = s(a \wedge b)$ iff $s(a \vee b) = s(b)$; hence*

$a/K = b/K$ iff $s(a) = s(a \wedge b) = s(b)$ iff $s(a) = s(a \vee b) = s(b)$. Moreover, the function $\hat{s}\colon A/K \to [0,1]$ defined by

$$\hat{s}(a/K) = s(a)$$

is a faithful state on $\mathbf{A}/K$.

Proof. Trivially, $0 \in K$ while $1 \notin K$. Recalling Lemma 4.4, it easily follows by Lemma 4.8 (ii), (iv) and (v) that K is an ideal of $\mathbf{A}$. We have $a/K \leq b/K$ iff $a \ominus b \in K$ iff $s(a \ominus b) = 0$. By Lemma 4.8 (iii) we conclude that this is equivalent to $s(a) = s(a \wedge b)$ as well as to $s(a \vee b) = s(b)$. Consequently, $a/K = b/K$ iff $s(a) = s(a \wedge b) = s(b)$ iff $s(a) = s(a \vee b) = s(b)$, which at once entails that the map $\hat{s}$ is well-defined. By Proposition 4.2, $\hat{s}$ is a state on $\mathbf{A}/K$. Indeed, $\hat{s}(1/K) = s(1) = 1$, and if $a/K \geq b/K$, then $s(a) = s(a \vee b)$ and $\hat{s}(a/K \ominus b/K) = s(a \ominus b) = s(a \vee b) - s(b) = s(a) - s(b) = \hat{s}(a/K) - \hat{s}(b/K)$. Actually, $\hat{s}$ is a faithful state because $\hat{s}(a/K) = 0$ iff $s(a) = 0 = s(0)$ iff $a/K = 0/K = K$. □

As usual, given $s\colon A \to [0,1]$, for any $\emptyset \neq X \subseteq A$ we write $s(X) = \{s(x) \mid x \in X\}$. Thus $\inf s(X)$ is $\inf\{s(x) \mid x \in X\}$.

Lemma 4.9. *Let $\mathbf{A}$ be a commutative basic algebra. For every ideal $I \in \mathcal{I}(\mathbf{A})$ and every state $s\colon A \to [0,1]$, the function $a/I \mapsto \inf s(a/I)$ is additive and monotone.*

Proof. Let us assume $a/I \leq \neg b/I$, i.e., $a/I = (a \wedge \neg b)/I$. We have

$$\inf s(a/I) + \inf s(b/I) = \inf\{s(a_1) + s(b_1) \mid a_1 \in a/I, b_1 \in b/I\}.$$

Since $s(a_1 \oplus b_1) \leq s(a_1) + s(b_1)$ and $a_1 \oplus b_1 \in (a \oplus b)/I$ for all $a_1 \in a/I$ and $b_1 \in b/I$, every element of the set $\{s(a_1) + s(b_1) \mid a_1 \in a/I, b_1 \in b/I\}$ dominates some element of $s(a/I \oplus b/I) = s((a \oplus b)/I)$, and it follows that $\inf s(a/I \oplus b/I) \leq \inf s(a/I) + \inf s(b/I)$.

Conversely, let $x \in (a \oplus b)/I$. By Lemma 4.1 (viii) we have $(x \ominus b) \oplus (b \wedge x) = x$ where $x \ominus b \leq 1 \ominus b = \neg b \leq \neg(b \wedge x)$, so that $s(x) = s(x \ominus b) + s(b \wedge x)$. But $x \ominus b \in ((a \oplus b) \ominus b)/I = (a \wedge \neg b)/I = a/I$ and $b \wedge x \in (b \wedge (a \oplus b))/I = b/I$.

Hence for every $x \in (a \oplus b)/I$, $s(x)$ may be written as $s(a_1) + s(b_1)$ where $a_1 \in a/I$ and $b_1 \in b/I$, thus $s(a/I \oplus b/I) \subseteq \{s(a_1) + s(b_1) \mid a_1 \in a/I, b_1 \in b/I\}$, which yields $\inf s(a/I \oplus b/I) \geq \inf s(a/I) + \inf s(b/I)$.

Altogether, we have proved that $\inf s(a/I \oplus b/I) = \inf s(a/I) + \inf s(b/I)$ provided $a/I \leq \neg b/I$. The monotonicity follows from the additivity because the arguments used in the proof of Lemma 4.8 (ii) rely on additivity only. $\square$

As a corollary we obtain:

Proposition 4.4. *If I is an ideal of a commutative basic algebra $\mathbf{A}$ and s is a state on $\mathbf{A}$ such that $\inf s(1/I) \neq 0$, then the map $s_I \colon A/I \to [0,1]$ defined by*

$$s_I(a/I) = \frac{\inf s(a/I)}{\inf s(1/I)}$$

is a state on A/I.

Proposition 4.5. *Let $\mathbf{A}$ be a commutative basic algebra. Let $I \in \mathcal{I}(\mathbf{A})$ be a proper non-zero ideal which is a polar in $\mathbf{A}$ (i.e. $I = I^{\perp\perp}$). Then for every faithful state s on $\mathbf{A}$ we have $\inf s(1/I) \neq 0$ and the function s_I defined in Proposition 4.4 is a faithful state on $\mathbf{A}/I$.*

Proof. First, we show that

$$\inf s(a/I) = s(a) \tag{4.6}$$

for $a \in I^\perp$. As a matter of fact, we show that a is the least element of a/I. Indeed, if $b \in a/I$, then $a/I = b/I$, so $a/I \ominus b/I = I$, which is equivalent to $a \ominus b \in I$. But we also have $a \ominus b \in I^\perp$ since $a \ominus b \leq a \in I^\perp$. Hence $a \ominus b \in I \cap I^\perp = \{0\}$ and $a \ominus b = 0$, which entails $a \leq b$. Since s is monotone, we have settled (4.6).

Second, we observe that

$$\inf s(a/I) = 0 \quad \Rightarrow \quad a \in I \tag{4.7}$$

for all $a \in A$. Suppose that $\inf s(a/I) = 0$, but $a \notin I = I^{\perp\perp}$, i.e., there exists $b \in I^\perp$ such that $a \wedge b \neq 0$. Then $a \wedge b \in I^\perp$, whence using (4.6) we get $0 = \inf s(a/I) \geq \inf s((a \wedge b)/I) = s(a \wedge b)$. Since s is a faithful state, this is possible only if $a \wedge b = 0$, but this is a contradiction.

Now, by (4.7) we have $\inf s(1/I) \neq 0$ for otherwise $1 \in I$, which would contradict the assumption $I \neq A$. Hence s_I is a faithful state on $\mathbf{A}/I$ because $Ker(s_I) = \{I\}$ owing to (4.7). $\square$

Theorem 4.5. *A representable commutative basic algebra which has a faithful state is an MV-algebra.*

Proof. Let $\mathbf{A}$ be a commutative basic algebra and $s\colon A \to [0,1]$ a faithful state on it. We aim at proving that $\mathbf{A}$ fulfills (4.2).

Let $a = (z \ominus y) \ominus (z \ominus x)$ and $b = x \ominus y$, and suppose by way of contradiction that the elements a, b are incomparable, i.e., $a \ominus b \neq 0 \neq b \ominus a$. Then $I = (a \ominus b)^{\perp}$ is a proper non-zero ideal since $a \ominus b \notin I$ and $b \ominus a \in I$. By Proposition 4.5, the function s_I is a faithful state on the quotient algebra $\mathbf{A}/I$ where we have $b/I \leq a/I$ as $b \ominus a \in I$, and so $s_I(a/I \ominus b/I) = s_I(a/I) - s_I(b/I)$. But $s_I(a/I) = s_I((z/I \ominus y/I) \ominus (z/I \ominus x/I)) \leq s_I(x/I \ominus y/I) = s_I(b/I)$ by Lemma 4.8 (v), and hence $s_I(a/I \ominus b/I) = 0$. Since s_I is a faithful state, it follows that $a/I \ominus b/I = I$, thus $a \ominus b \in I$. This is the desired contradiction.

We further know that $s(a) \leq s(b)$ by Lemma 4.8 (v). But by (vi) of the same lemma $a > b$ would yield $s(a) > s(b)$, thus the only possible case is $a \leq b$, proving that A is an MV-algebra. $\square$

Corollary 4.3. *Let $\mathbf{A}$ be a representable commutative basic algebra. For every state s on $\mathbf{A}$, $Ker(s)$ is an MV-ideal, i.e., the quotient algebra $\mathbf{A}/Ker(s)$ is an MV-algebra.*

Proof. Since $\hat{s}$ is a faithful state on $\mathbf{A}/Ker(s)$ by Proposition 4.3, it follows by Theorem 4.5 that $\mathbf{A}/Ker(s)$ is an MV-algebra. $\square$

4.3.2 State-morphisms

As in MV-algebras, we define a *state-morphism* on a commutative basic algebra $\mathbf{A}$ as a homomorphism of $\mathbf{A}$ to the standard MV-algebra $[0,1]_{MV}$

(recall that the operations in $[0,1]_{MV}$ are given by $x \oplus y = \min\{1, x+y\}$, $\neg x = 1 - x$ and $x \ominus y = \max\{0, x-y\}$).

Lemma 4.10. *Every state-morphism is a state. A state is a state-morphism if and only if it preserves finite suprema or, equivalently, finite infima.*

Proof. Let s be a state-morphism on $\mathbf{A}$. Clearly, $s(1) = 1$ and if $a \leq \neg b$, then $s(a) \leq s(\neg b) = 1 - s(b)$, so $s(a) + s(b) \leq 1$ and hence $s(a \oplus b) = s(a) \oplus s(b) = \min\{1, s(a) + s(b)\} = s(a) + s(b)$. This shows that s is a state.

For the latter statement, if s is a state preserving finite suprema, then $s(a \ominus b) = s(a \vee b) - s(b) = \max\{s(a), s(b)\} - s(b) = \max\{s(a) - s(b), 0\} = s(a) \ominus s(b)$. Similarly, if s preserves finite infima, then $s(a \ominus b) = s(a) - s(a \wedge b) = s(a) - \min\{s(a), s(b)\} = \max\{0, s(a) - s(b)\} = s(a) \ominus s(b)$. Thus s is a homomorphism. $\square$

Theorem 4.6. *Let $\mathbf{A}$ be a commutative basic algebra. A state $s\colon A \to [0,1]$ is a state-morphism if and only if its kernel $Ker(s)$ is an MV-ideal which is a maximal ideal. Every state-morphism is uniquely determined by its kernel.*

Proof. Let s be a state-morphism. Then $s(A)$ is a subalgebra of the standard MV-algebra $[0,1]_{MV}$ isomorphic to $\mathbf{A}/Ker(s)$, hence $Ker(s)$ is an MV-ideal. Since $s(\mathbf{A})$ is simple (up to isomorphism, simple MV-algebras are just the subalgebras of $[0,1]_{MV}$, cf. [46]), and since (by the Correspondence Theorem, e.g. [27], Theorem 6.20) the ideal lattice $\mathcal{I}(\mathbf{A}/Ker(s))$ of $\mathbf{A}/Ker(s)$ is isomorphic to the interval $[Ker(s), \mathbf{A}]$ in the ideal lattice $\mathcal{I}(\mathbf{A})$, it follows that $Ker(s)$ is a maximal ideal of $\mathbf{A}$.

Conversely, assume that $K = Ker(s)$ is an MV-ideal and that it is a maximal ideal. Then $\mathbf{A}/K$ is a simple MV-algebra, hence linearly ordered. For arbitrary $a, b \in A$, if $a/K \leq b/K$, then $s(a) = s(a \wedge b)$, and if $a/K \geq b/K$, then $s(a \wedge b) = s(b)$. In either case we have $s(a \wedge b) = \min\{s(a), s(b)\}$, thus by Lemma 4.10, s is a state-morphism.

For the latter claim, let h_1, h_2 be two state-morphisms on $\mathbf{A}$ with the same kernel, say K. Then both $h_1(A)$ and $h_2(A)$ are isomorphic to $\mathbf{A}/K$, and hence $h_1(A) \cong h_2(A)$, the isomorphism being the map $\eta\colon h_1(a) \mapsto$

$h_2(a)$. However, it is known that an isomorphism between two subalgebras of the standard MV-algebra $[0,1]_{MV}$ is automatically the identity map. Therefore $h_1(a) = h_2(a)$ for all $a \in A$. $\square$

4.3.3 State space

Given a commutative basic algebra $\mathbf{A}$, we let $\mathcal{S}(\mathbf{A})$ and $\mathcal{H}(\mathbf{A})$ denote the set of states and the set of state-morphisms on $\mathbf{A}$, respectively. We regard them as subspaces of the Tychonoff cube $[0,1]^A$, i.e., the set of all $[0,1]$-valued functions over $\mathbf{A}$ equipped with the usual product topology.

Theorem 4.7. *Let $\mathbf{A}$ be a commutative basic algebra. Then $\mathcal{S}(\mathbf{A})$ is either empty or a convex compact subset of $[0,1]^A$, and $\mathcal{H}(\mathbf{A})$ is either empty or a compact subset of $[0,1]^A$ and $\mathcal{S}(\mathbf{A})$ contains the closure of the convex hull of $\mathcal{H}(\mathbf{A})$. Moreover, if $\mathbf{A}$ is representable and $\mathcal{S}(\mathbf{A}) \neq \emptyset$, then the extreme boundary $\partial_e\mathcal{S}(\mathbf{A})$ of $\mathcal{S}(\mathbf{A})$ is exactly the set $\mathcal{H}(\mathbf{A})$ of state-morphisms.*

Proof. $\mathcal{S}(\mathbf{A})$ is convex: It can easily be seen that $\mathcal{S}(\mathbf{A})$ is closed under convex combinations: if $s_1, \dots, s_n$ are states and $\lambda_1, \dots, \lambda_n$ non-negative reals with $\sum_{i=1}^n \lambda_i = 1$, then also the function $\sum_{i=1}^n \lambda_i s_i$ is a state on $\mathbf{A}$.

$\mathcal{S}(\mathbf{A})$ is compact: Let $\{s_t \mid t \in T\}$ be a net in $\mathcal{S}(\mathbf{A})$ which converges to s in $[0,1]^A$. We have $s(1) = \lim_{t\in T} s_t(1) = 1$, and if $a \leq \neg b$, then $s(a \oplus b) = \lim_{t\in T} s_t(a \oplus b) = \lim_{t\in T}(s_t(a) + s_t(b)) = \lim_{t\in T} s_t(a) + \lim_{t\in T} s_t(b) = s(a) + s(b)$. Thus s is a state.

$\mathcal{H}(\mathbf{A})$ is compact: If $\{s_t \mid t \in T\}$ is a net of state-morphisms converging in $[0,1]^A$ to s, then $s(a \oplus b) = \lim_{t\in T} s_t(a \oplus b) = \lim_{t\in T}(s_t(a) \oplus s_t(b)) = \lim_{t\in T} s_t(a) \oplus \lim_{t\in T} s_t(b) = s(a) \oplus s(b)$ for all $a, b \in A$, so that s is a state-morphism.

Since $\mathcal{H}(\mathbf{A}) \subseteq \mathcal{S}(\mathbf{A})$, it immediately follows that the closure of the convex hull of $\mathcal{H}(\mathbf{A})$ is a subset of $\mathcal{S}(\mathbf{A})$, too.

Now, let $\mathbf{A}$ be a representable commutative basic algebra such that $\mathcal{S}(\mathbf{A}) \neq \emptyset$. The extreme boundary $\partial_e\mathcal{S}(\mathbf{A})$ consists of the *extremal states*, i.e., those $s \in \mathcal{S}(\mathbf{A})$ which cannot be written as a positive convex combination of two states distinct from s. We have to show that s is an extremal state iff it is a state-morphism.

Let $s \in \partial_e \mathcal{S}(\mathbf{A})$. By Proposition 4.3 and Corollary 4.3, we know that $\mathbf{A}/K$, where $K = Ker(s)$, is an MV-algebra and that the function $\hat{s}\colon a/K \mapsto s(a)$ is a state on $\mathbf{A}/K$. Suppose that $\hat{s}$ is a positive convex combination of states r_1, r_2 on $\mathbf{A}/K$, i.e., $\hat{s} = \lambda_1 r_1 + \lambda_2 r_2$ where $\lambda_1 + \lambda_2 = 1$ and $0 < \lambda_1, \lambda_2 < 1$. It is easily seen that the functions $s_i = \nu \circ r_i$, where $\nu\colon a \mapsto a/K$ is the natural homomorphism of $\mathbf{A}$ onto $\mathbf{A}/K$, are states on $\mathbf{A}$ such that $\hat{s}_i = r_i$ and $s = \lambda_1 s_1 + \lambda_2 s_2$. Then $s_1 = s = s_2$ since s is extremal. It follows that $r_1 = \hat{s} = r_2$, so $\hat{s}$ is an extremal state on $\mathbf{A}/K$. By [120], the extremal states on MV-algebras are precisely the state-morphisms. Hence $\hat{s}$ is a state-morphism on $\mathbf{A}/K$ and we conclude that s is a state-morphism on $\mathbf{A}$ since $s(a \oplus b) = \hat{s}((a \oplus b)/K) = \hat{s}(a/K) \oplus \hat{s}(b/K) = s(a) \oplus s(b)$.

Conversely, let $h \in \mathcal{H}(\mathbf{A})$ and assume that $h = \lambda_1 s_1 + \lambda_2 s_2$ for some $s_i \in \mathcal{S}(\mathbf{A})$ and $0 < \lambda_i < 1$. Then certainly $Ker(h) = Ker(s_1) \cap Ker(s_2)$. Since h is a state-morphism, $Ker(h)$ is a maximal ideal and hence $Ker(h) = Ker(s_1) = Ker(s_2)$. Then both s_1 and s_2 are state-morphisms, and since state-morphisms are uniquely determined by their kernels, it follows that $h = s_1 = s_2$. Thus $h \in \partial_e \mathcal{S}(\mathbf{A})$. $\square$

Since the Tychonoff cube $[0,1]^A$ is a locally convex Hausdorff space (i.e., the collection of all open convex subsets forms a basis) and $\mathcal{S}(\mathbf{A})$ is a compact convex subset of $[0,1]^A$, by the Krein-Mil'man Theorem (which says that a compact convex subset is the closure of the convex hull of its set of extreme points, see e.g. [106], Theorem 4.30) we get:

Corollary 4.4. *If* $\mathbf{A}$ *is a representable commutative basic algebra such that* $\mathcal{S}(\mathbf{A}) \neq \emptyset$*, then* $\mathcal{S}(\mathbf{A})$ *is the closure of the convex hull of* $\mathcal{H}(\mathbf{A})$*.*

In what follows, given a commutative basic algebra $\mathbf{A}$, we denote by $\mathcal{MMV}(\mathbf{A})$ the set of those MV-ideals of $\mathbf{A}$ which are maximal as ideals. We know that a state-morphism is specified by its kernel which belongs to $\mathcal{MMV}(\mathbf{A})$. Also conversely, to every $M \in \mathcal{MMV}(\mathbf{A})$ there corresponds a state-morphism. Indeed, $\mathbf{A}/M$ is a simple MV-algebra, so it is isomorphic to a subalgebra of $[0,1]_{MV}$. If ι is an embedding of $\mathbf{A}/M$ into $[0,1]_{MV}$ and ν the natural homomorphism of $\mathbf{A}$ onto $\mathbf{A}/M$, then $h = \nu \circ \iota$ is a state-morphism on $\mathbf{A}$ and $Ker(h) = M$. Therefore, the map $\eta\colon \mathcal{H}(\mathbf{A}) \to$

$\mathcal{MMV}(\mathbf{A})$ defined by

$$\eta(s) = Ker(s)$$

is a bijection. As a consequence, for representable commutative basic algebras we have: $\mathcal{S}(\mathbf{A}) \neq \emptyset$ iff $\mathcal{H}(\mathbf{A}) \neq \emptyset$ iff $\mathcal{MMV}(\mathbf{A}) \neq \emptyset$.

We now put a topology on $\mathcal{MMV}(\mathbf{A}) \neq \emptyset$ in the following standard way. For any $I \in \mathcal{I}(\mathbf{A})$, let

$$\mathcal{O}(I) = \{M \in \mathcal{MMV}(\mathbf{A}) \mid I \nsubseteq M\}.$$

For $a \in A$ and $M \in \mathcal{MMV}(\mathbf{A})$ we have $a \in M$ iff M contains the ideal generated by a, so we may unambiguously write $\mathcal{O}(a)$ for $\{M \in \mathcal{MMV}(\mathbf{A}) \mid a \notin M\}$. The following facts are obvious:

(i) $\mathcal{O}(\mathbf{A}) = \mathcal{MMV}(\mathbf{A})$ and $\mathcal{O}(0) = \emptyset$;

(ii) if I, J are ideals of $\mathbf{A}$, then $\mathcal{O}(I \cap J) = \mathcal{O}(I) \cap \mathcal{O}(J)$ (this follows from the fact that every maximal ideal is prime, i.e., $I \subseteq M$ or $J \subseteq M$ whenever $I \cap J \subseteq M$);

(iii) for every family $\{I_t \mid t \in T\}$ of ideals of $\mathbf{A}$, $\mathcal{O}(\bigvee_{t\in T} I_t) = \bigcup_{t\in T} \mathcal{O}(I_t)$, where $\bigvee_{t\in T} I_t$ is the join of the ideals I_t in the ideal lattice $\mathcal{I}(\mathbf{A})$.

Hence $\mathcal{T} = \{\mathcal{O}(I) \mid I \in \mathcal{I}(\mathbf{A})\}$ is a topology on $\mathcal{MMV}(\mathbf{A})$; its basis is the set $\{\mathcal{O}(a) \mid a \in A\}$ because if $M \in \mathcal{O}(I)$, then $I \nsubseteq M$ and for any $a \in I \setminus M$ we have $M \in \mathcal{O}(a) \subseteq \mathcal{O}(I)$.

Lemma 4.11. *Let $\mathbf{A}$ be a commutative basic algebra and $M \in \mathcal{MMV}(\mathbf{A})$. If $a \wedge b \in M$, then $a \in M$ or $b \in M$. Consequently, $\mathcal{O}(a) \cap \mathcal{O}(b) = \mathcal{O}(a \wedge b)$ for all $a, b \in A$.*

Proof. Let $a \wedge b \in M$ and suppose that $a \notin M$. Then certainly $MV(M \cup \{a\}) = \mathbf{A}$ since M is a maximal ideal strictly contained in $MV(M \cup \{a\})$, and hence, by Lemma 4.7, $b \leq x \oplus y$ where $x \in M$ and $y \in N(a)$. Since $a \wedge b \in M$ and $y = \tau(a, \ldots, a)$ for some additive term τ, applying Lemma 4.1 (v) repeatedly we obtain $y \wedge b \in M$. Consequently, $b \leq x \oplus y$ yields $b = (x \oplus y) \wedge b \leq (x \wedge b) \oplus (y \wedge b) \in M$ as both $x \wedge b$ and $y \wedge b$ belong to M. Thus $b \in M$. $\square$

Theorem 4.8. *For every commutative basic algebra* $\mathbf{A}$*, if* $\mathcal{MMV}(\mathbf{A}) \neq \emptyset$*, then* $\mathcal{MMV}(\mathbf{A})$ *equipped with the topology* $\mathcal{T}$ *is a compact Hausdorff space homeomorphic to* $\mathcal{H}(\mathbf{A})$*.*

Proof. Let $P, Q \in \mathcal{MMV}(\mathbf{A})$ be distinct, i.e., there exist $a \in P \setminus Q$ and $b \in Q \setminus P$. It is easily seen by Lemma 4.3 that $a \ominus b \in P \setminus Q$ and $b \ominus a \in Q \setminus P$. By Lemma 4.11 and owing to prelinearity (Lemma 4.1 (ix)) we have $\mathcal{O}(a \ominus b) \cap \mathcal{O}(b \ominus a) = \mathcal{O}((a \ominus b) \wedge (b \ominus a)) = \mathcal{O}(0) = \emptyset$, which means that $\mathcal{O}(b \ominus a)$ and $\mathcal{O}(a \ominus b)$ are disjoint open neighborhoods of P and Q, respectively. Thus $\mathcal{MMV}(\mathbf{A})$ is a Hausdorff space.

Further, recalling that $\eta\colon s \mapsto Ker(s)$ is a bijection of $\mathcal{H}(\mathbf{A})$ onto $\mathcal{MMV}(\mathbf{A})$, in order to complete the proof we only have to show that η is continuous. For that purpose, assume that $\mathcal{C}$ is a closed subset of $\mathcal{MMV}(\mathbf{A})$, i.e., $\mathcal{C} = \{M \in \mathcal{MMV}(\mathbf{A}) \mid I \subseteq M\}$ for some $I \in \mathcal{I}(\mathbf{A})$. For every state-morphism $s \in \mathcal{H}(\mathbf{A})$ we have: $\eta(s) \in \mathcal{C}$ iff $I \subseteq Ker(s)$ iff $s(x) = 0$ for all $x \in I$, hence

$$\eta^{-1}(\mathcal{C}) = \{s \in \mathcal{H}(\mathbf{A}) \mid s(x) = 0 \text{ for all } x \in I\},$$

which is a closed subset of $\mathcal{H}(\mathbf{A})$. Thus η is continuous. $\square$

Finally, we show that if a representable commutative basic algebra $\mathbf{A}$ admits a state, then the states on $\mathbf{A}$ correspond to the states on a certain MV-algebra. We will regard two elements of $\mathbf{A}$ as being equivalent iff we cannot separate them by a state, i.e., for $a, b \in A$,

$$a \sim b \quad \text{iff} \quad s(a) = s(b) \text{ for all } s \in \mathcal{S}(\mathbf{A}).$$

Since $\mathcal{S}(\mathbf{A})$ is the closure of the convex hull of $\mathcal{H}(\mathbf{A}) = \partial_e \mathcal{S}(\mathbf{A})$ (see Corollary 4.4), we have

$$a \sim b \quad \text{iff} \quad s(a) = s(b) \text{ for all } s \in \mathcal{H}(\mathbf{A}).$$

Hence $\sim$ is a congruence on $\mathbf{A}$, the 0-class of which is

$$K = \bigcap \{Ker(s) \mid s \in \mathcal{S}(\mathbf{A})\} = \bigcap \{Ker(s) \mid s \in \mathcal{H}(\mathbf{A})\}.$$

Recall that a homeomorphism is *affine* if it preserves convex combinations.

Theorem 4.9. *Let* $\mathbf{A}$ *be a representable commutative basic algebra. If* $\mathcal{S}(\mathbf{A}) \neq \emptyset$*, then the quotient algebra* $\mathbf{A}/K$ *is an MV-algebra such that the spaces* $\mathcal{S}(\mathbf{A}/K)$ *and* $\mathcal{H}(\mathbf{A}/K)$ *are affinely homeomorphic to* $\mathcal{S}(\mathbf{A})$ *and* $\mathcal{H}(\mathbf{A})$*, respectively.*

Proof. Clearly, $K = \bigcap\{M \mid M \in \mathcal{MMV}(\mathbf{A})\}$ is an MV-ideal, thus $\mathbf{A}/K$ is an MV-algebra. Let θ be the map assigning to every $s \in \mathcal{S}(\mathbf{A})$ the state $\hat{s} \in \mathcal{S}(\mathbf{A}/K)$ defined as in Proposition 4.3, i.e., $\hat{s}\colon a/K \mapsto s(a)$ for all $a \in A$. That $\hat{s}$ is well-defined follows from $K \subseteq Ker(s)$. We are going to prove that $\theta\colon s \mapsto \hat{s}$ is an affine homeomorphism of $\mathcal{S}(\mathbf{A})$ onto $\mathcal{S}(\mathbf{A}/K)$. Let ν be the natural homomorphism of $\mathbf{A}$ onto $\mathbf{A}/K$. Then for every $m \in \mathcal{S}(\mathbf{A}/K)$, the composite map $\nu \circ m\colon a \mapsto m(a/K)$ is a state on $\mathbf{A}$ such that $\widehat{\nu \circ m} = m$, and every $s \in \mathcal{S}(\mathbf{A})$ can be recovered from $\hat{s}$ as $\nu \circ \hat{s}$. Thus $\theta\colon s \mapsto \hat{s}$ is a bijection between $\mathcal{S}(\mathbf{A})$ and $\mathcal{S}(\mathbf{A}/K)$. It is readily seen that θ also preserves limits of nets and convex combinations, so it is an affine homeomorphism as claimed. $\square$

4.3.4 De Finetti's coherent maps and Borel states

Let now $\mathbf{A}$ be a non-empty set and $\mathcal{V}$ be a non-empty closed subset of the cube $[0,1]^A$. A function $s\colon A \to [0,1]$ is said to satisfy *de Finetti's coherence criterion* with respect to $\mathcal{V}$ (see [113], [121]) if for every $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, there exists $v \in \mathcal{V}$ such that

$$\sum_{i=1}^{n} \lambda_i(s(a_i) - v(a_i)) \geq 0. \tag{4.8}$$

This property can be interpreted in a quite natural way (see [121] for more details): Let us think of $a_1, \ldots, a_n \in A$ as certain "events". Then the maps $v\colon A \to [0,1]$ are "valuations" assigning to each a_i its "truth-value" $v(a_i) \in [0,1]$. If we think of $s(a_i) \in [0,1]$ as bookmaker's "belief" that a_i will take place, and λ_i as bettor's "stake" (the bettor pays the bookmaker $\sum_{i=1}^{n} \lambda_i s(a_i)$ and will receive $\sum_{i=1}^{n} \lambda_i v(a_i)$ when $v(a_i)$'s are revealed), then the condition (4.8) essentially says that there are no "stakes" $\lambda_1, \ldots, \lambda_n$ ensuring the player to win in every case.

Generalizing the classical de Finetti's no-Dutch-Book theorem, Mundici [121] proved that if $\mathbf{A}$ is a free MV-algebra and $\mathcal{V} = \mathcal{H}(\mathbf{A})$ is the set of homomorphisms to the standard MV-algebra $[0,1]_{MV}$, then the coherent maps with respect to $\mathcal{V}$ are precisely the states (i.e. the functions in the closure of the convex hull of $\mathcal{V}$). More generally, it was proved in [113] that for any non-empty set $\mathbf{A}$ and any non-empty closed subset $\mathcal{V}$ of $[0,1]^A$ (without any algebraic requirements), a map $s\colon A \to [0,1]$ is coherent with respect to $\mathcal{V}$ if and only if s belongs to the closure of the convex hull of $\mathcal{V}$. The theorem for MV-algebras (resp. for the Łukasiewicz logic) then follows immediately, and Corollary 4.4 entails that the results of [113] are also applicable to representable commutative basic algebras. In particular, we get the following:

Theorem 4.10. *Let* $\mathbf{A}$ *be a representable commutative basic algebra. Then for every function* $s\colon A \to [0,1]$ *the following are equivalent:*

(i) *s is a state;*

(ii) *s is coherent with respect to the set $\mathcal{H}(\mathbf{A})$ of all state-morphisms on $\mathbf{A}$;*

(iii) *s belongs to the closure of the convex hull of $\mathcal{H}(\mathbf{A})$, i.e., s is the limit of a net of convex combinations of state-morphisms.*

As an example of "non-associative events" illustrating that sometimes things are commutative and associativity does not make much sense, we can take inheritance of blood types. One's blood type is determined by the parents' blood types, and this "conjunction" is commutative. This is one of those situations mentioned in Section 1 (cf. [83]) where there is no need to consider associativity, but if we move from parents to grandparents and ask about the possible blood types of their grandchildren, then non-associativity comes out. Indeed, if the grandparents are (A and 0) and (B and 0), then the grandchildren may have any of the four blood types, while if the grandparents were (A and B) and (0 and 0), then the type AB would be impossible.

Finally, recalling another result from [113], we tie the states on a commutative basic algebra $\mathbf{A}$ to the Borel probability measures on $\mathcal{H}(\mathbf{A})$. Let

A and $\mathcal{V}$ be as above a non-empty set and a non-empty closed subset of $[0,1]^A$, respectively. By [113], Theorem 4.2, the coherent maps with respect to $\mathcal{V}$ coincide with the so-called *Borel states* on **A**, i.e., those maps $s\colon A \to [0,1]$ for which there is a regular Borel probability measure μ on $\mathcal{V}$ such that

$$s(a) = \int_{\mathcal{V}} f_a \,\mathrm{d}\mu \tag{4.9}$$

for all $a \in A$, where $f_a\colon \mathcal{V} \to [0,1]$ is given by $f_a(v) = v(a)$ for all $v \in \mathcal{V}$.

Therefore, by Corollary 4.4, we have:

Corollary 4.5. *Let* **A** *be a representable commutative basic algebra. Then* $s\colon A \to [0,1]$ *is a state on* **A** *if and only if* s *is a Borel state, i.e., it satisfies* (4.9) *for some regular Borel probability measure* μ *on* $\mathcal{H}(\mathbf{A})$.

This is a generalization of Panti's theorem [126] which asserts that for any MV-algebra **A**, the states on **A** correspond to the regular Borel probability measures on the space of maximal ideals of **A** (which is homeomorphic to $\mathcal{H}(\mathbf{A})$, cf. Theorem 4.8), as well as of Kroupa's theorem [112] which asserts the same for semisimple MV-algebras.

4.4 Non-associative generalization of Hájek's BL-algebras

4.4.1 Non-associative residuated lattices

The following lemma extends well known properties of residuated lattices to a non-associative case.

Lemma 4.12. *If* $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *is a non-associative residuated lattice then for all* $x, y, x_1, x_2 \in A$ *we have*

(i) $x \leq y$ *if and only if* $x \rightarrow y = 1$,

(ii) If $x_1 \leq x_2$ *then* $x_1 \cdot y \leq x_2 \cdot y$, $x_2 \rightarrow y \leq x_1 \rightarrow y$ *and* $y \rightarrow x_1 \leq y \rightarrow x_2$,

(iii) $y \cdot (x_1 \vee x_2) = (y \cdot x_1) \vee (y \cdot x_2)$,

(iv) $y \rightarrow (x_1 \wedge x_2) = (y \rightarrow x_1) \wedge (y \rightarrow x_2)$,

(v) $(x_1 \vee x_2) \rightarrow y = (x_1 \rightarrow y) \wedge (x_2 \rightarrow y)$,

(vi) $(x \rightarrow y) \cdot x \leq x, y$,

(vii) $(x \rightarrow y) \rightarrow y \geq x, y$.

Proof. (i) Clearly, the adjointness property yields $1 \cdot x \leq y$ if and only if $1 \leq x \rightarrow y$.
(ii),(vi),(vii) Assume $x_1 \leq x_2$. Then $y \cdot x_2 = y \cdot x_2$ yields $x_2 \leq y \rightarrow (y \cdot x_2)$ and consequently $x_1 \leq x_2 \leq y \rightarrow (y \cdot x_2)$. Again, due to the adjointness property we obtain $y \cdot x_1 \leq y \cdot x_2$.

Further, $x \rightarrow y \leq x \rightarrow y$ gives $(x \rightarrow y) \cdot x \leq y$ and $x \leq (x \rightarrow y) \rightarrow y$.

Due to $x \rightarrow y \leq 1$ and monotonicity of "$\cdot$" we give $(x \rightarrow y) \cdot x \leq 1 \cdot x = x$ which verifes *(vi)*.

As $y \cdot (x \rightarrow y) \leq y \cdot 1 = y$, we conclude $y \leq (x \rightarrow y) \rightarrow y$ which gives *(vii)*.

By the adjointness property, $(y \rightarrow x_1) \cdot y \leq x_1 \leq x_2$ imply $y \rightarrow x_1 \leq y \rightarrow x_2$.

Analogously, $x_1 \leq x_2 \leq (x_2 \to y) \to y$ yield $x_1 \cdot (x_2 \to y) \leq y$ and finally we obtain $x_2 \to y \leq x_1 \to y$.
(iii) Due to monotonicity of "$\cdot$" and $x_1 \vee x_2 \geq x_1, x_2$ we obtain $(x_1 \vee x_2) \cdot y \geq x_1 \cdot y, x_2 \cdot y$ and $(x_1 \vee x_2) \cdot y \geq (x_1 \cdot y) \vee (x_2 \cdot y)$.

From *(ii)* we conclude $x_1 \leq y \to (x_1 \cdot y) \leq y \to ((x_1 \cdot y) \vee (x_2 \cdot y))$ and analogously $x_2 \leq y \to (x_2 \cdot y) \leq y \to ((x_1 \cdot y) \vee (x_2 \cdot y))$ which give $x_1 \vee x_2 \leq y \to ((x_1 \cdot y) \vee (x_2 \cdot y))$. Applying the adjointness property gives $(x_1 \vee x_2) \cdot y \leq (x_1 \cdot y) \vee (x_2 \cdot y)$.
(iv) Clearly, $x_1 \wedge x_2 \leq x_1, x_2$ implies $y \to (x_1 \wedge x_2) \leq y \to x_1, y \to x_2$ and $y \to (x_1 \wedge x_2) \leq (y \to x_1) \wedge (y \to x_2)$.

We have shown $x_1 \geq y \cdot (y \to x_1) \geq y \cdot ((y \to x_1) \wedge (y \to x_2))$ and $x_2 \geq y \cdot (y \to x_2) \geq y \cdot ((y \to x_1) \wedge (y \to x_2))$. Altogether we obtain $x_1 \wedge x_2 \geq y \cdot ((y \to x_1) \wedge (y \to x_2))$ and finally $y \to (x_1 \wedge x_2) \geq (y \to x_1) \wedge (y \to x_2)$.
(v) We have $(x_1 \vee x_2) \to y \leq x_1 \to y, x_2 \to y$ which gives $(x_1 \vee x_2) \to y \leq (x_1 \to y) \wedge (x_2 \to y)$.

The inequalities

$$x_1 \leq (x_1 \to y) \to y \leq ((x_1 \to y) \wedge (x_2 \to y)) \to y,$$

$$x_2 \leq (x_2 \to y) \to y \leq ((x_1 \to y) \wedge (x_2 \to y)) \to y$$

yield

$$x_1 \vee x_2 \leq ((x_1 \to y) \wedge (x_2 \to y)) \to y.$$

Due to adjointness property we obtain

$$(x_1 \vee x_2) \cdot ((x_1 \to y) \wedge (x_2 \to y)) \leq y,$$

and finally

$$(x_1 \to y) \wedge (x_2 \to y) \leq (x_1 \vee x_2) \to y.$$

□

As a corollary we obtain

Corollary 4.6. *Let* $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ *be a non-associative residuated lattice. If* $x, y \in A$ *then* *(i)* $1 \to x = x$,

(ii) $x \to y = (x \vee y) \to y$,

(iii) *If* $x \vee y = 1$ *then* $x \to y = y$ *holds.*

Proof. *(i)* Clearly, $1 \to x = 1 \cdot (1 \to x) \leq x$ holds. Conversely, $1 \cdot x \leq x$ yields $x \leq 1 \to x$.

(ii) Applying Lemma 4.12(v) we compute $(x \vee y) \to y = (x \to y) \wedge (y \to y) = (x \to y) \wedge 1 = x \to y$.

(iii) If $x \vee y = 1$ then $x \to y = (x \vee y) \to y = 1 \to y = y$. □

We remark that the non-associative residuated lattice $\mathbf{A}$ is *1-regular* if and only if for any congruences $\theta_1, \theta_2 \in Con\ \mathbf{A}$ is $1/\theta_1 = 1/\theta_2 \iff \theta_1 = \theta_2$ (see [34]).

Theorem 4.11. *The class of all non-associative residuated lattices forms an arithmetical variety. Moreover, the variety is* 1*-regular.*

Proof. To show that the class of all non-associative residuated lattices forms a variety we replace (A3) of Definition 5.3 (which is in fact a couple of quasi-identities) by the following identities (of course, the inequalities can be expressed by identities as we have a lattice structure) :

(M1) $x \to y \leq x \to (y \vee z)$,

(M2) $x \leq y \to (x \cdot y)$,

(M3) $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$

(M4) $x \cdot (x \to y) \leq y$.

It is easy to show (due to Lemma 4.12) that (M1)-(M4) hold in non-associative residuated lattices. Conversely, we will show that (M1)-(M4) yield the adjointness property

Assume $x \cdot y \leq z$. Then the equality $(x \cdot y) \vee z = z$ holds and thus $y \to ((x \cdot y) \vee z) = y \to z$. Applying to (M1) and (M2) we compute $y \leq y \to (x \cdot y) \leq y \to ((x \cdot y) \vee z) = y \to z$.

Conversely, if $x \leq y \to z$ hold then (M3) and (M4) yield $x \cdot y \leq (x \cdot y) \vee ((y \to z) \cdot y) = (x \vee (y \to z)) \cdot y = (y \to z) \cdot y \leq z$.

To prove the arithmeticity, it is enough to find a Pixley term $m(x, y, z)$, i.e. a term verifying the identities $m(y, y, x) = m(x, y, x) = m(x, y, y) = x$. One can easily check that term $m(x, y, z) := (x \cdot ((x \wedge y) \to z)) \vee (z \cdot ((y \wedge z) \to x))$ is just the Pixley term for the variety of all non-associative residuated lattices.

It is obvious that $x \to y \wedge y \to x = 1$ if and only if $x = y$. This fact yields 1-regularity of the variety (see [34]).

□

4.4.2 Congruences and filters

In this section we describe congruence kernels (called filters) in the variety of non-associative residuated lattices. In what follows, we need the unary terms

$$\alpha_b^a(x) := (a \cdot b) \to (a \cdot (b \cdot x)),$$
$$\beta_b^a(x) := b \to (a \to ((a \cdot b) \cdot x)).$$

The following lemma justifies their importance in non-associtive residuated lattices.

Lemma 4.13. *If* $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ *is a non-associative residuated lattice then for all* $a, b, c \in A$ *we have:*

(i) $(a \cdot b) \cdot \alpha_b^a(x) \leq a \cdot (b \cdot x)$,

(ii) $a \cdot (b \cdot \beta_b^a(x)) \leq (a \cdot b) \cdot x$.

Proof. Due to Lemma 4.12(vi) we obtain

$$(a \cdot b) \cdot \alpha_b^a(x) = (a \cdot b) \cdot ((a \cdot b) \to (a \cdot (b \cdot x))) \leq a \cdot (b \cdot x).$$

Analogously, we compute

$$a \cdot (b \cdot \beta_b^a(x)) = a \cdot (b \cdot (b \to (a \to ((a \cdot b) \cdot x)))) \leq a \cdot (a \to ((a \cdot b) \cdot x)) \leq (a \cdot b) \cdot x.$$

□

Definition 4.1. *Let* $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *be a non-associative residuated lattice. A non-empty subset* $F \subseteq A$ *is called a* **filter** *of the non-associative residuated lattice* $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *if it satisfies:*

(F1) If $x \in F$ *and* $y \in A$ *such that* $x \leq y$ *then* $y \in F$.

(F2) $x \cdot y \in F$ *for all* $x, y \in F$.

(F2) If $a, b \in A$ *and* $x \in F$ *then* $\alpha_b^a(x), \beta_b^a(x) \in F$.

Clearly, due to Corollary 4.6 we obtain $\alpha_b^a(1) = \beta_b^a(1) = 1$ for all $a, b \in A$. Thus, it can be easily proved that for any non-associative residuated lattice $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ and any congruence θ on $\mathbf{A}$, the class $1/\theta$ is a filter (in the sense of previous definition). Conversely, we can state:

Theorem 4.12. *Let* $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *be a non-associative residuated lattice and let* F *be a filter. Then the relation*

$$\Theta(F) = \{(x, y) \in A^2 \mid x \rightarrow y, y \rightarrow x \in F\}$$

is a congruence on $\mathbf{A}$*. Moreover,* $1/\Theta(F) = F$.

Proof. Reflexivity and symmetry. Clearly, $x \rightarrow x = 1$ thus $(x, x) \in \Theta(F)$ for all $x \in A$. Symmetry is easily seen by definition.

Transitivity. If $x \rightarrow y, y \rightarrow z \in F$ then by Definition 4.1 we obtain $(x \rightarrow y) \cdot \beta^x_{x \rightarrow y}(y \rightarrow z) \in F$. Moreover, Lemma 4.13(ii) shows that $x \cdot ((x \rightarrow y) \cdot \beta^x_{x \rightarrow y}(y \rightarrow z)) \leq (x \cdot (x \rightarrow y)) \cdot (y \rightarrow z) \leq y \cdot (y \rightarrow z) \leq z$. Due to adjointness property we obtain $(x \rightarrow y) \cdot \beta^x_{x \rightarrow y}(y \rightarrow z) \leq x \rightarrow z$ and finally $x \rightarrow z \in F$.

We have proved that $x \rightarrow y, y \rightarrow z \in F$ yields $x \rightarrow z \in F$. Similarly, we can prove $z \rightarrow x \in F$ which gives transitivity of $\Theta(F)$.

Further, let $(x_1, y_1), (x_2, y_2) \in \Theta(F)$, thus $x_1 \rightarrow y_1, x_2 \rightarrow y_2 \in F$. We prove the compatibility of $\Theta(F)$ with respect to $\wedge, \vee, \cdot$ and $\rightarrow$:

Compatibility of $\wedge$. We have $(x_1 \wedge x_2) \rightarrow (y_1 \wedge y_2) = ((x_1 \wedge x_2) \rightarrow y_1) \wedge ((x_1 \wedge x_2) \rightarrow y_2) \geq (x_1 \rightarrow y_1) \wedge (x_2 \rightarrow y_2)$. Clearly, $(x_1 \rightarrow y_1) \wedge (x_2 \rightarrow y_2) \in F$ and consequently $(x_1 \wedge x_2) \rightarrow (y_1 \wedge y_2) \in F$. Analogously, we can prove that $(y_1 \wedge y_2) \to (x_1 \wedge x_2) \in F$ and thus $(x_1 \wedge x_2, y_1 \wedge y_2) \in \Theta(F)$ holds.

Compatibility of $\vee$. We compute $(x_1 \vee x_2) \to (y_1 \vee y_2) = (x_1 \to (y_1 \vee y_2)) \wedge (x_2 \to (y_1 \vee y_2)) \geq (x_1 \to y_1) \wedge (x_2 \to y_2)$. Clearly, $(x_1 \to y_1) \wedge (x_2 \to y_2) \in F$ and thus also $(x_1 \vee x_2) \to (y_1 \vee y_2) \in F$. Similarly, we prove $(y_1 \vee y_2) \to (x_1 \vee x_2) \in F$ which verifies $(x_1 \vee x_2, y_1 \vee y_2) \in \Theta(F)$.

Compatibility of $\cdot$. We have

$$\alpha^{x_1}_{x_2}((x_2 \to y_2) \cdot \beta^{x_2}_{x_2 \to y_2}(\beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1)))) \in F.$$

Due to Lemma 4.13 we compute

$$\begin{aligned}
& (x_1 \cdot x_2) \cdot \alpha^{x_1}_{x_2}((x_2 \to y_2) \cdot \beta^{x_2}_{x_2 \to y_2}(\beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1)))) \\
\leq\ & x_1 \cdot (x_2 \cdot ((x_2 \to y_2) \cdot \beta^{x_2}_{x_2 \to y_2}(\beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1))))) \\
\leq\ & x_1 \cdot ((x_2 \cdot (x_2 \to y_2))\beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1))) \\
\leq\ & x_1 \cdot (y_2 \cdot \beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1))) \\
\leq\ & (x_1 \cdot y_2) \cdot \alpha^{y_2}_{x_1}(x_1 \to y_1) \\
\leq\ & y_2 \cdot (x_1 \cdot (x_1 \to y_1)) \\
\leq\ & y_1 \cdot y_2.
\end{aligned}$$

The last inequality shows that

$$\alpha^{x_1}_{x_2}((x_2 \to y_2) \cdot \beta^{x_2}_{x_2 \to y_2}(\beta^{x_1}_{y_2}(\alpha^{y_2}_{x_1}(x_1 \to y_1)))) \leq (x_1 \cdot x_2) \to (y_1 \cdot y_2)$$

and verifies $(x_1 \cdot x_2) \to (y_1 \cdot y_2) \in F$. Analogously, we prove that also $(y_1 \cdot y_2) \to (x_1 \cdot x_2) \in F$ which shows $(x_1 \cdot x_2, y_1 \cdot y_2) \in \Theta(F)$.

Compatibility of $\to$. The proof of the last part we split to three steps.

- *We will prove that* $(x, y) \in \Theta(F)$ *yields* $(z \to x, z \to y) \in \Theta(F)$. Indeed, if $x \to y \in F$ then also $\beta^{z}_{z \to x}(x \to y) \in F$. Applying Lemma 2(ii) we obtain

$$z \cdot ((z \to x) \cdot \beta^{z}_{z \to x}(x \to y)) \leq (z \cdot (z \to x)) \cdot (x \to y) \leq x \cdot (x \to y) \leq y.$$

Due to adjointness property we conclude

$$\beta^z_{z\to x}(x \to y) \leq (z \to x) \to (z \to y)$$

and $(z \to x) \to (z \to y) \in F$. Analogously, we prove that $(z \to y) \to (z \to x) \in F$ and hence $(z \to x, z \to y) \in \Theta(F)$.

- *We will prove that* $(x, y) \in \Theta(F)$ *yields* $(x \to z, y \to z) \in \Theta(F)$. As above, $x \to y \in F$ gives $\beta^x_{y\to z}(\alpha^{y\to z}_x(x \to y)) \in F$ and

$$\begin{aligned} x \cdot ((y \to z) \cdot \beta^x_{y\to z}(\alpha^{y\to z}_x(x \to y))) &\leq (x \cdot (y \to z)) \cdot \alpha^{y\to z}_x(x \to y) \\ &\leq (y \to z) \cdot (x \cdot (x \to y)) \\ &\leq (y \to z) \cdot y \\ &\leq z. \end{aligned}$$

By using of adjointness property we obtain

$$\beta^x_{y\to z}(\alpha^{y\to z}_x(x \to y)) \leq (y \to z) \to (x \to z).$$

Thus $(y \to z) \to (x \to z) \in F$ and analogously $(x \to z) \to (y \to z) \in F$ which gives $(x \to z, y \to z) \in \Theta(F)$.

Due to the above claims, we have $(x_1 \to x_2, y_1 \to x_2), (y_1 \to x_2, y_1 \to y_2) \in \Theta(F)$. Finally, transitivity of $\Theta(F)$ yields $(x_1 \to x_2, y_1 \to y_2) \in \Theta(F)$. □

We have seen that the variety of all non-associative residuated lattices is 1-regular. One can easily prove that the correspondences $F \longmapsto \Theta(F)$ and $\Theta \longmapsto 1/\Theta$ between filters and congruences are mutually inverse.

4.4.3 Representable non-associative residuated lattices

Let us denote by $\Re$ the sub-variety of the variety of non-associative residuated lattices generated just by its linearly ordered members and called it representable.

In what follows we need the following notation: for any $M \subseteq A$ denote $M^{\perp} = \{x \in A \mid x \vee y = 1 \text{ (for all } y \in M)\}$. Moreover, we introduce the identities

$$(x \to y) \vee \alpha_b^a(y \to x) = 1 \qquad (\alpha\text{-prelinearity}),$$

$$(x \to y) \vee \beta_b^a(y \to x) = 1 \qquad (\beta\text{-prelinearity}).$$

Lemma 4.14. *Let* $\mathbf{A} \in \Re$ *be a representable non-associative residuated lattice. Then for all* $x, y, z \in A$ *we have*

(i) $x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z)$,

(ii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(iii) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Proof. Since $\mathbf{A}$ is representable, it suffices to prove that (i)-(iii) are valid for all linearly ordered members. So assume that $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ is linearly ordered non-associative residuatted lattice. If $x, y, z \in A$ then due to linearity we may assume $y \leq z$. Then due to Lemma 4.12 we obtain the inequalities $x \cdot y \leq x \cdot z$, $x \to y \leq x \to z$ and $y \to x \leq z \to x$. Altogether, we obtain

$$x \cdot (y \wedge z) = x \cdot y = (x \cdot y) \wedge (x \cdot z),$$

$$x \to (y \vee z) = x \to z = (x \to y) \vee (x \to z),$$

$$(x \wedge y) \to z = (x \to z) \vee (y \to z).$$

□

Lemma 4.15. *Every non-associative residuated lattice with the divisibility law satisfies the Riesz decomposition property, i.e. given* $x, y, z \in A$ *with* $x \cdot y \leq z$ *then there are the elements* $x' \geq x$ *and* $y' \geq y$ *such that* $x' \cdot y' = z$ *holds.*

Proof. Denote $x' = y \to z$ and $y' = (y \to z) \to z$. Adjointness property and Lemma 4.12 yield $x' \geq x$ and $y' \geq y$. Due to divisibility we obtain

$$x' \cdot y' = (y \to z) \cdot ((y \to z) \to z) = (y \to z) \wedge z = z.$$

□

Lemma 4.16. *The following conditions are equivalent:*

(i) $\mathbf{A} \in \Re$

(ii) $\mathbf{A}$ *is a non-associative residuated lattice with α-prelinearity and β-prelinearity.*

(iii) $\mathbf{A}$ *is a non-associative residuated lattice with prelinearity and satisfying the quasi-identities*

$$x \vee y = 1 \implies x \vee \alpha_b^a(y) = 1 \text{ and } x \vee \beta_b^a(y) = 1 \qquad (P)$$

(iv) $\mathbf{A}$ *is non-associative residuated lattice with prelinearity and, for all $M \subseteq A$, the set $M^\perp$ is a filter of* $\mathbf{A}$.

(v) $\mathbf{A}$ *is a subdirect product of linearly ordered non-associative residuated lattices.*

Proof. (i) ⇒ (ii). Again, it suffices to verify (ii) for linearly ordered non-associative residuated lattices. Clearly, in this case we have for any $x, y \in A$ either $x \leq y$ or $y \leq x$, thus $x \to y = 1$ or $y \to x = 1$. Evidently, $\mathbf{A}$ satisfies prelinearity. As $\alpha_b^a(1) = 1 = \beta_b^a(1)$ for all $a, b \in A$, $\mathbf{A}$ verifies α-prelinearity and β-prelinearity.

(ii) ⇒ (iii). Of course, $\alpha_1^1(y \to x) = x \to y$ and consequently α-prelinearity yields prelinearity. If $x \vee y = 1$ holds, then $x \vee y = 1$, hence $x \to y = y, y \to x = x$. Applying α-prelinerity we obtain $1 = (y \to x) \vee \alpha_b^a(x \to y) = x \vee \alpha_b^a(y)$. This shows $x \vee \alpha_b^a(y) = 1$. Analogously it can be proved $x \vee \beta_b^a(y) = 1$.

(iii) ⇒ (iv). The set $M^\perp$ is evidently an order filter on A. If $x, y \in M^\perp$ then we have $x \vee m = y \vee m = 1$ for all $m \in M$, thus $(x \cdot y) \vee m =$

$(x \cdot y) \vee (x \cdot m) \vee m = x \cdot (y \vee m) \vee m = (x \cdot 1) \vee m = 1$. Altogether, $x \cdot y$ belongs to $M^\perp$ and $M^\perp$ is closed on products. Finally, assume $x \in M^\perp$, $a, b \in A$ and $m \in M$. Then $m \vee x = 1$ and due to *(iii)* we obtain $m \vee \alpha_b^a(x) = 1$ and $m \vee \beta_b^a(x) = 1$. This yields $\alpha_b^a(x), \beta_b^a(x) \in M^\perp$ and shows that $M^\perp$ is a filter of $\mathbf{A}$.

(iv) $\Rightarrow$ *(v).* First let $\mathbf{B}$ be a subdirectly irreducible algebra which satisfies *(iv)*. Thus $M^\perp$ is a filter of $\mathbf{B}$ for all $M \subseteq B$. We shall show that $\mathbf{B}$ is linearly ordered. Assume to the contrary that there are $x, y \in B$ with $x \parallel y$, i.e. $x \to y \neq 1$ and $y \to x \neq 1$. Applying prelinearity of $\mathbf{B}$ we have $(x \to y) \vee (y \to x) = 1$. Thus the sets $\{x \to y\}^\perp, \{x \to y\}^{\perp\perp}$ are filters of $\mathbf{B}$. Clearly, both are non-trivial since $y \to x \in \{x \to y\}^\perp$, $x \to y \notin \{x \to y\}^\perp$, $x \to y \in \{x \to y\}^{\perp\perp}$ and $y \to x \notin \{x \to y\}^{\perp\perp}$. Moreover, $\{x \to y\}^\perp \cap \{x \to y\}^{\perp\perp} = \{1\}$. Due to 1-regularity of the variety we obtain $\Theta(\{x \to y\}^\perp) \cap \Theta(\{x \to y\}^{\perp\perp}) = \Delta$ which shows subdirect reducibility of $\mathbf{B}$, a contradiction.

We have proved that any subdirectly irreducible algebra with *(iv)* is linearly ordered and since the class of those algebras forms a variety, any algebra $\mathbf{A}$ with it (iv) is subdirect product of chains.

(v) $\Rightarrow$ *(i).* Any subdirect product of linearly ordered algebras belongs to $\Re$. □

Corollary 4.7. *A non-associative residuated lattice* $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ *is representable if it satisfies* α*-prelinearity and* β*-prelinearity.*

Definition 4.2. *An algebra* $\mathbf{A} \in \Re$ *is called a non-associative BL algebra (naBL algebra briefly) if it satisfies divisibility.*

Theorem 4.13. *A non-associative residuated lattice is an naBL algebra if it satisfies divisibility,* α*-prelinearity and* β*-prelinearity.*

Proof. Directly follows from Lemma 4.16. □

Lemma 4.17. *Every naBL algebra* $\mathbf{A} \in \Re$ *satisfies the identities* $a \cdot (b \cdot x) = (a \cdot b) \cdot \alpha_b^a(x)$ *and* $(a \cdot b) \cdot x = a \cdot (b \cdot \beta_b^a(x))$ *for* $a, b, x \in A$.

Proof. Using Lemma 4.14 and divisibility we compute $(a \cdot b) \cdot \alpha_b^a(x) = (a \cdot b) \cdot ((a \cdot b) \to (a \cdot (b \cdot x))) = (a \cdot b) \wedge (a \cdot (b \cdot x)) = a \cdot (b \cdot x)$. Similarly, $a \cdot (b \cdot \beta_b^a(x)) = a \cdot (b \cdot (b \to (a \to ((a \cdot b) \cdot x)))) = a \cdot (b \wedge (a \to ((a \cdot b) \cdot x))) = (a \cdot b) \wedge (a \cdot (a \to ((a \cdot b) \cdot x))) = (a \cdot b) \wedge a \wedge ((a \cdot b) \cdot x) = (a \cdot b) \cdot x$. □

Theorem 4.14. *Let $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ be a an naBL-algebra. Let $F \subseteq A$ be a non-empty subset which is closed on upper bounds and on the product. Then F is a filter of $\mathbf{A}$ if and only if $a \cdot (b \cdot F) = (a \cdot b) \cdot F$ for all $a, b \in A$.*

Proof. Let F be a filter of $\mathbf{A}$. If $f \in a \cdot (b \cdot F)$ then there is $x \in F$ such that $f = a \cdot (b \cdot x) = (a \cdot b) \cdot \alpha_b^a(x)$. As $\alpha_b^a(x) \in F$, we conclude $f \in (a \cdot b) \cdot F$. Conversely, any $f \in (a \cdot b) \cdot F$ can be written in the form $f = (a \cdot b) \cdot x = a \cdot (b \cdot \beta_b^a(x))$, where $x \in F$, and thus also $\beta_b^a(x) \in F$. Altogether, we obtain $f \in a \cdot (b \cdot F)$.

Assume that the equality $a \cdot (b \cdot F) = (a \cdot b) \cdot F$ holds. For all $a, b \in A$ and $x \in F$ there is $x' \in F$ such that $a \cdot (b \cdot x) = (a \cdot b) \cdot x'$. Thus $x' \leq (a \cdot b) \to (a \cdot (b \cdot x)) = \alpha_b^a(x)$ which yields $\alpha_b^a(x) \in F$. Analogously, for all $a, b \in A$ and $x \in F$ there is $x' \in F$ such that $(a \cdot b) \cdot x = a \cdot (b \cdot x')$, which gives $x' \leq b \to (a \to ((a \cdot b) \cdot x)) = \beta_b^a(x)$. We have proved $\beta_b^a(x) \in F$. □

Definition 4.3. *A binary operation '$*$' on the interval $[0, 1]$ of reals is said to be a non-associative t-norm (nat-norm briefly) if*

- *(nat1)* $([0, 1]; *, 1)$ *is a commutative groupoid with the neutral element* 1.
- *(nat2) The operation $*$ is continuous as a function, in the usual interval topology on $[0, 1]^2$.*
- *(nat3) If $x, y, z \in [0, 1]$ are such that $x \leq y$ then $x \cdot z \leq y \cdot z$.*

Theorem 4.15. *For any nat-norm there is a unique '$\to_*$' satisfying the adjointness property, i.e., $x * y \leq z$ if and only if $x \leq y \to_* z$. Moreover, an algebra $([0, 1], \vee, \wedge, *, \to_*, 0, 1)$, where $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$, is an naBL algebra.*

Proof. At first we prove that the operation '$\to_*$' is correctly defined. For any $x \in [0,1]$ consider the function $\phi_x : [0,1] \longrightarrow [0,1]$ defined by $\phi_x(y) = x * y$. The continuity of '$*$' yields the continuity of ϕ_x. Denote $M = \{z \in [0,1] \mid z * x \leq y\} = \{z \in [0,1] \mid \phi_x(z) \leq y\}$. Then due to the continuity of ϕ_x we obtain $y \geq \bigvee_{z \in M}(\phi_x(z)) = \phi_x(\bigvee M)$ which gives $\bigvee M \in M$. Thus there is $\max\{z \in [0,1] \mid z * x \leq y\} = \bigvee\{z \in [0,1] \mid z * x \leq y\}$. We put $x \to_* y := \max\{z \in [0,1] \mid z * x \leq y\}$.

If $x * y \leq z$ then $x \leq \max\{a \in [0,1] \mid a * y \leq z\} = y \to_* z$. Conversely, if $x \leq y \to_* z = \max\{a \in [0,1] \mid a * y \leq z\}$ then due to continuity of '$*$' we have $x*y \leq \max\{a \in [0,1] \mid a*y \leq z\}*y = \max\{a*y \in [0,1] \mid a*y \leq z\} \leq z$. This proves adjointness condition. If there is another operation '$\to_o$' with the adjointness condition $x * y \leq z$ if and only if $x \leq y \to_o z$, then clearly $a \leq x \to_* y$ if and only if $a \leq x \to_o y$ and thus $\to_* = \to_o$.

We have proved that the algebra $([0,1]; \vee, \wedge, *, \to_*, 0, 1)$ is a linearly ordered non-associative residuated groupoid. Finally, we prove the divisibility. Take $x, y \in [0,1]$ with $y \leq x$. Then clearly $\phi_x(1) = x \geq y \geq 0 = \phi_x(0)$ and the continuity and monotonicity of ϕ_x yield the existence of $a \in [0,1]$ such that $a*x = \phi_x(a) = y$. Now $a \leq x \to_* y$, and $y = a*x \leq (x \to_* y) \leq y$. Then $(x \to_* y) * x = y = x \wedge y$. Finally, if $x \leq y$ then clearly $x \to_* y = 1$ and $(x \to_* y) * x = 1 * x = x = x \wedge y$.

□

4.4.4 The main theorem

The class of all naBL algebras will be denoted by $na\mathcal{BL}$ and denote by $na\mathcal{T}$ the class of all $naBL$ algebras induced just by na$t$ norms. In what follows we show that $na\mathcal{T}$ is the generating class for the variety $na\mathcal{BL}$.

Theorem 4.16. *(main theorem)* $na\mathcal{BL} = \mathrm{IP_S SP_U}(na\mathcal{T})$.

Proof. Take any linearly ordered $naBL$ algebra $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$. We prove that $\mathbf{A}$ fulfils general finite embedding property for the class $na\mathcal{T}$. Let $X \subseteq A$ be any finite set such that $0, 1 \in X$. We put $X \cdot X = \{x \cdot y \mid x \in X,\ y \in X\}$. Clearly, $1 \in X$ yields the inequality $X \subseteq X \cdot X$. Finiteness of X yields finiteness of $X \cdot X$ add thus we may assume that $X \cdot X = \{x_0, \cdots, x_n\}$

and $X = \{y_0, \cdots, y_m\}$ where $0 = x_0 < x_1 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < \cdots < y_m$. Introduce the mapping $f : X \cdot X \longrightarrow [0,1]$ by $f(x_i) = \frac{i}{n}$. The mapping f is a lattice embedding. For any $y_i, y_j \in X$ the product $y_i \cdot y_j$ belongs to $X \cdot X$. Consequently, for any $y_i, y_j \in \{y_0, \cdots, y_{n-1}\}$, we can define the function $\varphi[y_i, y_j] : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\varphi[y_i, y_j](x, y) =$$

$$\frac{f(y_i \cdot y_j) + f(y_{i+1} \cdot y_{j+1}) - f(y_{i+1} \cdot y_j) - f(y_i \cdot y_{j+1})}{(f(y_i) - f(y_{i+1}))(f(y_j) - f(y_{j+1}))}(x - f(y_i))(y - f(y_j)) +$$

$$\frac{f(y_i \cdot y_j) - f(y_{i+1} \cdot y_j)}{f(y_i) - f(y_{i+1})}(x - f(y_i)) +$$

$$\frac{f(y_i \cdot y_j) - f(y_i \cdot y_{j+1})}{f(y_j) - f(y_{j+1})}(y - f(y_j)) + f(y_i \cdot y_j).$$

The graph of the function $\varphi[y_i, y_j]$ formes a hypar incident with the points $(f(y_i), f(y_j), f(y_i \cdot y_j))$, $(f(y_{i+1}), f(y_j), f(y_{i+1} \cdot y_j))$, $(f(y_i), f(y_{j+1}), f(y_i \cdot y_{j+1}))$ and $(f(y_{i+1}), f(y_{j+1}), f(y_{i+1} \cdot y_{j+1}))$.

Clearly, the commutativity of '$\cdot$' induces

$$\varphi[y_i, y_j](x, y) = \varphi[y_j, y_i](y, x). \qquad (I)$$

Moreover, one can easily check that

$$\varphi[y_i, y_j](x, f(y_j)) = \varphi[y_i, y_{j-1}](x, f(y_j)). \qquad (II)$$

Point out that $y_m = 1$ and thus $f(y_m) = 1$. This yield

$$\varphi[y_i, y_{m-1}](x, f(y_m)) = \varphi[y_i, y_{m-1}](x, 1) = x. \qquad (III)$$

Also the equality

$$\varphi[y_i, y_j](f(y_i), f(y_j)) = f(y_i \cdot y_j) \qquad (IV)$$

can be easily verified.

The interval $[0,1]$ of reals can be splitted to the subintervals $I_1 = [f(y_0), f(y_1)]$, $I_2 = [f(y_1), f(y_2)]$, $\cdots$, $I_m = [f(y_{m-1}), 1]$.

Further, introduce the operation '$*$' such that for any $x \in I_i$ and $y \in I_j$ we put

$$x * y = \varphi[y_i, y_j](x, y).$$

The condition (I) yields the commutativity of '$*$'. Clearly, commutativity of '$*$' and (II) prove that the functions $\varphi[y_i, y_j]$ have the same value on the common border of any two intervals $I_i \times I_j$ and $I_{i+1} \times I_j$ (or $I_i \times I_j$ and $I_i \times I_{j+1}$). Consequently, the function '$*$' is also continuous as it is continuously glued by hypars. Now we prove the following claim:

Claim 1. *The function $\varphi[y_i, y_j](x, y)$ for any $y_i, y_j \in \{y_0, \cdots, y_{n-1}\}$ and for a fixed $x \in I_i$ is non-decreasing (thus if $y_1, y_2 \in \mathbb{R}$ such that $y_1 \leq y_2$ then $\varphi[y_i, y_j](x, y_1) \leq \varphi[y_i, y_j](x, y_2)$).*

Proof. For any $y_i, y_j \in \{y_0, \cdots, y_{n-1}\}$ and any fixed $x \in I_i$ define the function

$$g(y) := \varphi[y_i, y_j](x, y).$$

Evidently, g is linear the angular coefficient

$$C = \frac{f(y_i \cdot y_j) + f(y_{i+1} \cdot y_{j+1}) - f(y_{i+1} \cdot y_j) - f(y_i \cdot y_{j+1})}{(f(y_i) - f(y_{i+1}))(f(y_j) - f(y_{j+1}))}(x - f(y_i))$$

$$+\frac{f(y_i \cdot y_j) - f(y_{i+1} \cdot y_j)}{f(y_i) - f(y_{i+1})}.$$

Putting $a := f(y_i \cdot y_j) - f(y_i \cdot y_{j+1})$, $b := f(y_{i+1} \cdot y_{j+1}) - f(y_{i+1} \cdot y_j)$ and $c := \frac{x - f(y_i)}{f(y_i) - f(y_{-+1})}$ we have

$$C = \frac{1}{f(y_j) - f(y_{j+1})}(bc + a(c+1)).$$

Moreover, we obtain

- $y_i \cdot y_j \leq y_i \cdot y_{j+1}$ yield $a \leq 0$,
- $y_{i+1} \cdot y_{j+1} \geq y_{i+1} \cdot y_j$ yield $b \geq 0$,

- $x \in I_i$ yield $f(y_i) \leq x \leq f(y_{i+1})$ and thus $0 \geq c \geq -1$ (and $0 \leq c+1 \leq 1$).
- $y_j < y_{j+1}$ yield $\frac{1}{f(y_j)-f(y_{j+1})} < 0$

Previous arguments show that $C \geq 0$ and thus the function g is non-decreasing. □

Claim 1 implies the monotonicity of '$*$'. Further, $0 * 0 = \varphi[0,0](0,0) = 0$. Finally, due to (III) we obtain $x * 1 = \varphi[y_i, y_{m-1}](x, 1) = x$ for any $x \in I_i$. Thus 1 is a neutral element of '$*$'and its monotonicity show that $0 = 0 * 0 \leq x * y \leq 1 * 1 = 1$. Altogether, '$*$' is an operation on the interval $[0,1]$ and which is the na$t$ norm.

Due to Theorem 4.15 we have a $naBL$ algebra $([0,1], \vee, \wedge, *, \rightarrow_*, 0, 1)$ which belongs to the class $na\mathcal{T}$.

Finally, we prove that the mapping $f : X \longrightarrow [0,1]$ is an embedding from the partial algebra $\mathbf{A}|_{\mathbf{X}}$ to $([0,1]; \vee, \wedge, *, \rightarrow_*, 0, 1)$. The mapping f is injective and monotone, thus it is a lattice embedding with $f(0) = 0$ and $f(1) = 1$.

Take any $y_i, y_j \in X$. Then $f(y_i) * f(y_j) = \varphi[y_i, y_j](f(y_i), f(y_j)) = f(y_i \cdot y_j)$. Thus the mapping f preserves the operation '$*$'. It suffices to show that f preserves also $\rightarrow$:

Claim 2. *Let $y_i, y_j \in X$ be such that $y_i \rightarrow y_j \in X$. Then $f(y_i \rightarrow y_j) = f(y_i) \rightarrow_* f(y_j)$.*

Proof. If $y_i \leq y_j$ then $f(y_i \rightarrow y_j) = f(1) = 1 = f(y_i) \rightarrow_* f(y_j)$.

Assume further $y_i > y_j$. Due to $y_i \rightarrow y_j \in X$ we obtain that $y_i \rightarrow y_j = y_k \neq 1$ for some $y_k \in X$.

As we have proved before, we obtain $f(y_i) * f(y_i \rightarrow y_j) = f(y_i \cdot (y_i \rightarrow y_j)) = f(y_i \wedge y_j) = f(y_j)$. Thus $f(y_i \rightarrow y_j) \leq f(y_i) \rightarrow_* f(y_j)$.

We prove that $y_i \cdot y_k < y_i \cdot y_{k+1}$ (in case $y_i \cdot y_{k+1} \leq y_i \cdot y_k = y_i \cdot (y_i \rightarrow y_j) = y_i \wedge y_j = y_j$ we would get $y_{k+1} \leq y_i \rightarrow y_j = y_k$, which is contradiction).

The function $h(x) = f(y_i) * x$ is linear on the interval I_{k+1} and $h(f(y_k)) = f(y_i) * f(y_k) = f(y_i \cdot y_k) < f(y_i \cdot y_{k+1}) = f(y_i) * f(y_{k+1}) = h(f(y_{k+1}))$, showing that it is strictly increasing on it. Note that $I_{k+1} = [f(y_k), f(y_{k+1})]$ and altogether we obtain for any $x \in [0,1]$ the property

$$f(y_k) < x \text{ implies } h(f(y_k)) < h(x). \qquad (*)$$

Finally, assume by contradiction the inequality $f(y_k) = f(y_i \to y_j) < f(y_i) \to_* f(y_j)$. The condition $(*)$ yields $h(f(y_k)) < h(f(y_i) \to_* f(y_j))$ which gives $f(y_j) = f(y_i \wedge y_j) = f(y_i \cdot (y_i \to y_j)) = f(y_i \cdot y_k) = f(y_i) * f(y_k) = h(f(y_k)) < h(f(y_i) \to_* f(y_j)) = f(y_i) * (f(y_i) \to_* f(y_j)) = f(y_i) \wedge f(y_j) = f(y_j)$, a contradiction. □

Altogether we have shown that $\mathbf{A}$ (being linearly ordered) satisfies general finite embedding property for the class $na\mathcal{T}$. By Theorem 1.3 we conclude that any linearly ordered $naBL$ algebra belongs to the class $\mathrm{ISP_U}(na\mathcal{T})$ and since any subdirectly irreducible $naBL$ algebra is linearly ordered (see Theorem 4.16), the proof is finished. □

Moreover, the main theorem has some important corollaries:

Corollary 4.8. *Any subdirectly irreducible $naBL$ algebra belongs to the class* $\mathrm{ISP_U}(na\mathcal{T})$.

Denoting for a class $\mathcal{X}$ of algebras of the same type $\mathcal{V}(\mathcal{X})$ the variety generated by $\mathcal{X}$ and $\mathcal{QV}(\mathcal{X})$ the quasivariety generated by $\mathcal{X}$, we have

Corollary 4.9. $na\mathcal{BL} = \mathcal{V}(na\mathcal{T}) = \mathcal{QV}(na\mathcal{T})$.

Proof. The proof follows from the previous corollary and the fact that varieties and quasivarieties are closed on direct products, subalgebras and ultraproducts. □

Chapter 5

State morphisms on algebras

Recently, Flaminio and Montagna in [82] presented an algebraizable logic containing probabilistic reasoning, and its equivalent algebraic semantic is the variety of state MV-algebras. We recall that a *state MV-algebra* is an MV-algebra whose language is extended adding an operator, τ (called also an *internal state*), whose properties are inspired by the ones of states. The analogues of extremal states are *state-morphism operators*, introduced in [53]. By definition, it is an idempotent endomorphism on an MV-algebra.

State MV-algebras generalize, for example, Hájek's approach, [97], to fuzzy logic with modality Pr (interpreted as *probably*) which has the following semantic interpretation: The probability of an event a is presented as the truth value of $\Pr(a)$. On the other hand, if s is a state, then $s(a)$ is interpreted as averaging of appearing the many valued event a.

We note that if $(\mathbf{M}, \tau)$ is a state MV-algebra, assuming that the range $\tau(\mathbf{M})$ is a simple MV-algebra, we see that it is a subalgebra of the real interval $[0,1]$ and therefore, τ can be regarded as a standard state on $\mathbf{M}$. On the other hand, every MV-algebra $\mathbf{M}$ can be embedded into the tensor product $[0,1] \otimes \mathbf{M}$, therefore, given a state s on $\mathbf{M}$, we define an operator τ_s on $[0,1] \otimes \mathbf{M}$ via $\tau_s(t \otimes a) := t \cdot s(a)$, [82, Thm 5.3]. Then due to [53, Thm 3.2], τ_s is a state-operator that is a state-morphism operator iff s is an extremal state. Thus, there is a natural correspondence between the notion of a state and an extremal state on one side, and a state-operator and a state-morphism operator on the other side.

Subdirectly irreducible state-morphism MV-algebras were described in [53, 62] and this was extended also for state-morphism BL-algebras in [65]. A complete description of both subdirectly irreducible state MV-algebras as well as subdirectly irreducible state-morphism MV-algebras can be found in [72]. In [61], it was shown that if $(\mathbf{M},\tau)$ is a state MV-algebra whose image $\tau(\mathbf{M})$ belongs to the variety generated by the $L_1,\ldots,L_n$, where $L_i := \{0,1/i,\ldots,i/i\}$, then τ has to be a state-morphism operator. The same is true if $\mathbf{M}$ is linearly ordered, [53]. Recently, in [72], we have shown that the unit square $[0,1]^2$ with the diagonal operator generates the whole variety of state-morphism MV-algebras; it answered in positive an open problem posed in [53]. In addition, there was shown that in contrast to MV-algebras, the lattice of subvarieties is uncountable. Moreover, it was shown that every subdirectly irreducible state-morphism MV-algebra can be embedded into some diagonal one.

We continue in the study of state BL-algebras and state-morphism BL-algebras. Because the methods developed in [72] are so general that, it is possible to study more general structures than MV-algebras or BL-algebras under a common umbrella. Hence, we introduce state-morphism algebras $(\mathbf{A},\tau)$, where the algebra $\mathbf{A}$ is an arbitrary algebra of type F and τ is an idempotent endomorphism of $\mathbf{A}$, i.e., a retract morphism $\tau : A \to \tau(A)$. Then general results applied to special types of algebras give interesting new results.

The main goals are:

(1) Characterizations of subdirectly irreducible state BL-algebras and state-morphism BL-algebras.

(2) Showing that every subdirectly state-morphism algebra can be embedded into some diagonal one $D(\mathbf{B}) := (\mathbf{B}\times\mathbf{B},\tau_B)$, where $\tau(a,b) = (a,a)$, $a,b \in B$, which is also subdirectly irreducible.

(3) We show that if $\mathcal{K}$ is a generator of some variety $\mathcal{V}$ of algebras of type F, then the system of diagonal state-morphism algebras $\{D(\mathbf{B}) \mid \mathbf{B} \in \mathcal{K}\}$ is a generator of the variety of state-morphism algebras whose F-reduct belongs to $\mathcal{V}$.

(4) We exhibit cases when the Congruence Extension Property holds for a variety of state-morphism algebras.

(5) In particular, a generator of the variety of state-morphism BL-algebras is the class of all BL-algebras of the real interval $[0, 1]$ equipped with a continuous t-norm. Similarly, a generator of the variety of state-morphism MTL-algebras is the class of all MTL-algebras of the real interval equipped with a left-continuous t-norm, similarly for non-associative BL-algebras one is the set of all non-associative BL-algebras of the real interval $[0, 1]$ equipped with a non-associative t-norm, and a generator of the variety of state-morphism pseudo MV-algebras is any pseudo MV-algebra $\Gamma(G, u)$, where (G, u) is a doubly transitive unital ℓ-group.

5.1 Subdirectly Irreducible State BL-algebras

In this section, we define state BL-algebras and state-morphism BL-algebras and we present a complete description of their subdirectly irreducible algebras. These results generalize those from [53, 62, 65, 72].

Let $\mathbf{M} = (M; \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra. For any $a \in M$, we define a complement $a^- := a \to 0$. Then $a \le a^{--}$ for any $a \in M$ and a BL-algebra is an MV-algebra iff $a^{--} = a$ for any $a \in M$.

A non-empty set $F \subseteq M$ is called a *filter* of $\mathbf{M}$ (or a *BL-filter* of $\mathbf{M}$) if for every $x, y \in M$: (1) $x, y \in F$ implies $x \odot y \in F$, and (2) $x \in F$, $x \le y$ implies $y \in F$. A filter $F \ne M$ is called a *maximal filter* if it is not strictly contained in any other filter $F' \ne M$. A BL-algebra is called *local* if it has a unique maximal filter.

We denote by $\mathrm{Rad}_1(\mathbf{M})$ the intersection of all maximal filters of $\mathbf{M}$.

Let $\mathbf{M}$ be a BL-algebra. A mapping $\tau : M \to M$ such that, for all $x, y \in M$, we have

$(1)_{BL}$ $\tau(0) = 0$;

$(2)_{BL}$ $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;

$(3)_{BL}$ $\tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y))$;

$(4)_{BL}$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;

$(5)_{BL}$ $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$

is said to be a *state-operator* on $\mathbf{M}$, and the pair $(\mathbf{M}, \tau)$ is said to be a *state BL-algebra*, or more precisely, a *BL-algebra with internal state*.

If $\tau : M \to M$ is a BL-endomorphism such that $\tau \circ \tau = \tau$, then τ is a state-operator on $\mathbf{M}$ and it is said to be a *state-morphism operator* and the couple $(\mathbf{M}, \tau)$ is said to be a *state-morphism BL-algebra*.

A filter F of a BL-algebra $\mathbf{M}$ is said to be a τ-filter if $\tau(F) \subseteq F$. If τ is a state-operator on $\mathbf{M}$, we denote by

$$\mathrm{Ker}(\tau) = \{a \in M \mid \tau(a) = 1\}.$$

then $\mathrm{Ker}(\tau)$ is a τ-filter. A state-operator τ is said to be *faithful* if $\mathrm{Ker}(\tau) = \{1\}$.

We recall that there is a one-to-one relation between congruences and τ-filters on a state BL-algebra $(\mathbf{M}, \tau)$ as follows. If F is a τ-filter, then the relation $\sim_F$ given by $x \sim_F y$ iff $x \to y, y \to x \in F$ is a congruence of the BL-algebra $\mathbf{M}$ and $\sim_F$ is also a congruence of the state BL-algebra $(\mathbf{M}, \tau)$.

Conversely, let $\sim$ be a congruence of state BL-algebra $(\mathbf{M}, \tau)$ and set $F_\sim := \{x \in M \mid x \sim 1\}$. Then $F_\sim$ is a τ-filter of $(\mathbf{M}, \tau)$ and $\sim_{F_\sim} = \sim$ and $F = F_{\sim_F}$.

By [51, Lemma 3.5(k)], $(\tau(\mathbf{M}), \tau)$ is a subalgebra of $(\mathbf{M}, \tau)$, τ on $\tau(M)$ is the identity, and hence, $(\mathrm{Ker}(\tau); \to, 0, 1)$ is a subhoop of $\mathbf{M}$. We say that two subhoops, A and B, of a BL-algebra $\mathbf{M}$ have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

Nevertheless a subdirectly irreducible state BL-algebra $(\mathbf{M}, \tau)$ is not necessarily linearly ordered, according to [51, Thm 5.5], $\tau(\mathbf{M})$ is always linearly ordered.

We note that according to [51, Prop 3.13], if $\mathbf{M}$ is an MV-algebra, then the notion of a state MV-algebra coincides with the notion of a state BL-algebra.

The following three characterizations were originally proved in [72] for state MV-algebras. Here we show that the original proofs from [72] slightly improved work also for state BL-algebras.

Lemma 5.1. *Suppose that* $(\mathbf{M}, \tau)$ *is a state BL-algebra. Then:*

(1) *If* τ *is faithful, then* $(\mathbf{M}, \tau)$ *is a subdirectly irreducible state BL-algebra if and only if* $\tau(\mathbf{M})$ *is a subdirectly irreducible BL-algebra.*

Now let $(\mathbf{M}, \tau)$ *be subdirectly irreducible. Then:*

(2) $\mathrm{Ker}(\tau)$ *is (either trivial or) a subdirectly irreducible hoop.*

(3) $\mathrm{Ker}(\tau)$ *and* $\tau(\mathbf{M})$ *have the disjunction property.*

Proof. (1) Suppose τ is faithful. Let F denote the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. There are two cases: (i) If $\tau(M) \cap F \neq \{1\}$, then $\tau(M) \cap F$ is the least nontrivial filter of $\tau(\mathbf{M})$ and $\tau(\mathbf{M})$ is subdirectly irreducible. (ii) If $\tau(\mathbf{M}) \cap F = \{1\}$, then for all $x \in F$, $\tau(x) = 1$ because $\tau(x) \in \tau(M) \cap F$ and $F \subseteq \mathrm{Ker}(\tau) = \{1\}$ is the trivial filter, a contradiction. Therefore, only the first case is possible and $\tau(\mathbf{M})$ is subdirectly irreducible.

Conversely, let $\tau(\mathbf{M})$ be subdirectly irreducible and let G be the least nontrivial filter of $\tau(\mathbf{M})$. Then the τ-filter F of $(\mathbf{M}, \tau)$ generated by G is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. Indeed, if K is another nontrivial τ-filter of $(\mathbf{M}, \tau)$, then $K \cap \tau(M) \supseteq F \cap \tau(M) = G$. Then K contains the τ-filter generated by G, that is $F \subseteq K$ which proves F is the least and $(\mathbf{M}, \tau)$ is subdirectly irreducible.

Now let $(\mathbf{M}, \tau)$ be subdirectly irreducible and let F denote the least nontrivial filter of $(\mathbf{M}, \tau)$.

(2) Suppose that τ is not faithful. Then $\mathrm{Ker}(\tau)$ is a nontrivial τ-filter. If $(\mathbf{M}, \tau)$ is subdirectly irreducible, it has a least nontrivial τ-filter, F say. So, $F \subseteq \mathrm{Ker}(\tau)$, and hence F is the least nontrivial filter of the hoop $\mathrm{Ker}(\tau)$. Hence, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible hoop.

(3) Suppose, by way of contradiction, that for some $x \in \mathrm{Ker}(\tau)$ and $y = \tau(y) \in \tau(M)$ one has $x < 1$, $y < 1$ and $x \vee y = 1$. It is easy to see that the BL-filters generated by x and by y, respectively, are τ-filters. Therefore they both contain F. Hence, the intersection of these filters contains F. Now let $c < 1$ be in F. Then there is a natural number n such that $x^n \leq c$ and $y^n \leq c$. It follows that $1 = (x \vee y)^n = x^n \vee y^n \leq c$, a contradiction. $\square$

Lemma 5.2. *If* $(\mathbf{M}, \tau)$ *is a subdirectly irreducible state BL-algebra, then* $\tau(M)$ *and* $\mathrm{Ker}(\tau)$ *are linearly ordered.*

Proof. According to [51, Thm 5.5], $\tau(M)$ is always linearly ordered. On the other hand, by Lemma 5.1, $\mathrm{Ker}(\tau)$ is either a trivial hoop or a subdirectly irreducible hoop, and hence it is linearly ordered. □

Theorem 5.1. *Let* $(\mathbf{M}, \tau)$ *be a state BL-algebra satisfying conditions* (1), (2) *and* (3) *in Lemma* 5.1*. Then* $(\mathbf{M}, \tau)$ *is subdirectly irreducible.*

Proof. Suppose first that τ is faithful and that $\tau(\mathbf{M})$ is subdirectly irreducible. Let F_0 be the least nontrivial filter of $\tau(\mathbf{M})$ and let F be the BL-filter of $\mathbf{M}$ generated by F_0. Then F is a τ-filter. Indeed, if $x \in F$, then there is $\tau(a) \in F_0$ and a natural number n such that $\tau(a)^n \leq x$. It follows that $\tau(x) \geq \tau(\tau(a)^n) = \tau(a)^n$, and $\tau(x) \in F$.

We assert that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. First of all, $\tau(\mathbf{M})$, being a subdirectly irreducible BL-algebra, is linearly ordered. Now in order to prove that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$, it suffices to prove that every τ-filter G not containing F is trivial. Now let $c < 1$ in $F \backslash G$. Then since $\mathrm{Ker}(\tau) = \{1\}$, $\tau(c) < 1$. Next, let $d \in G$. Then $\tau(d) \in G$, and for every n it cannot be $\tau(d)^n \leq \tau(c)$, otherwise $\tau(c) \in G$. Hence, for every n, $\tau(c) < \tau(d)^n$, and hence $\tau(c)$ does not belong to the τ-filter of $\tau(\mathbf{M})$ generated by $\tau(d)$. By the minimality of F in $\tau(\mathbf{M})$, $\tau(d) = 1$ and since τ is faithful, we conclude that $d = 1$ and G is trivial, as desired.

Now suppose that $\mathrm{Ker}(\tau)$ is nontrivial. By condition (2), $\mathrm{Ker}(\tau)$ is subdirectly irreducible. Thus, let F be the least nontrivial filter of $\mathrm{Ker}(\tau)$. Then F is a non trivial τ-filter, and we have to prove that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. Let G be any non trivial τ-filter of $(\mathbf{M}, \tau)$. If $G \subseteq \mathrm{Ker}(\tau)$, then it contains the least filter, F, of $\mathrm{Ker}(\tau)$, and $F \subseteq G$. Otherwise, G contains some $x \notin \mathrm{Ker}(\tau)$, and hence it contains $\tau(x) < 1$. Now by the disjunction property, for all $y < 1$ in $\mathrm{Ker}(\tau)$, $\tau(x) \vee y < 1$ and $\tau(x) \vee y \in \mathrm{Ker}(\tau) \cap G$. Thus, G contains the filter generated by $\tau(x) \vee y$, which is a non trivial filter of the hoop $\mathrm{Ker}(\tau)$, and hence it contains F, the least nontrivial filter of $\mathrm{Ker}(\tau)$. This proves the claim. □

By [72, Thm 3.5], conditions (1), (2), and (3) from Lemma 5.1 are independent ones even for state BL-algebras. In addition, Theorem 5.1 gives a characterization of subdirectly irreducible state BL-algebras. If $(\mathbf{M}, \tau)$ is a state-morphism BL-algebra, combining [65, Thm 4.5] we can say more about subdirectly irreducible state-morphism BL-algebras. The following examples are from [65].

Example 5.1. Let $\mathbf{M}$ be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

$$\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M. \tag{2.0}$$

Then τ_1 and τ_2 are two state-morphism operators on $\mathbf{M} \times \mathbf{M}$. Moreover, $(\mathbf{M} \times \mathbf{M}, \tau_1)$ and $(\mathbf{M} \times \mathbf{M}, \tau_2)$ are isomorphic state BL-algebras under the isomorphism $(a,b) \mapsto (b,a)$.

We say that an element $a \in M$ is *Boolean* if $a^{--} = a$ and $a \odot a = a$. Let $B(\mathbf{M})$ be the set of Boolean elements. Then $0, 1 \in B(\mathbf{M})$, $B(\mathbf{M})$ is a subset of the MV-skeleton $\mathrm{MV}(\mathbf{M}) := \{x \in M \mid x^{--} = x\}$, and $a \in B(\mathbf{M})$ implies $a^- \in B(\mathbf{M})$. We recall that according to [143, Thm 2], $\mathrm{MV}(\mathbf{M})$ is an MV-algebra, therefore, $B(\mathbf{M})$ is a Boolean subalgebra of $\mathrm{MV}(\mathbf{M})$.

Example 5.2. Let $\mathbf{B}$ be a local MV-algebra such that $\mathrm{Rad}_1(\mathbf{B}) \neq \{1\}$ is a unique nontrivial filter of B. Let $\mathbf{M}$ be a subalgebra of $\mathbf{B} \times \mathbf{B}$ that is generated by $\mathrm{Rad}_1(\mathbf{B}) \times \mathrm{Rad}_1(\mathbf{B})$, that is $M = (\mathrm{Rad}_1(\mathbf{B}) \times \mathrm{Rad}_1(\mathbf{B})) \cup (\mathrm{Rad}_1(\mathbf{B}) \times \mathrm{Rad}_1(\mathbf{B}))^-$. Let $\tau(x,y) := (x,x)$ for all $x, y \in M$. Then τ is a state-morphism operator on $\mathbf{M}$, $\mathrm{Ker}(\tau) = \{1\} \times \mathrm{Rad}_1(\mathbf{B}) \subset \mathrm{Rad}_1(\mathbf{M}) = \mathrm{Rad}_1(\mathbf{B}) \times \mathrm{Rad}_1(\mathbf{B})$, $\mathbf{M}$ has no Boolean nontrivial elements, and $(\mathbf{M}, \tau)$ is a subdirectly irreducible state-morphism MV-algebra that is not linear.

Example 5.3. Let $\mathbf{A}$ be a linear nontrivial BL-algebra and $\mathbf{B}$ a nontrivial subdirectly irreducible BL-algebra with the smallest nontrivial BL-filter F_B and let $h: \mathbf{A} \to \mathbf{B}$ be a BL-homomorphism. On $M = \mathbf{A} \times \mathbf{B}$ we define a mapping $\tau_h: M \to M$ by

$$\tau_h(a,b) = (a, h(a)), \quad (a,b) \in M. \tag{2.2}$$

If we set $y = (0, 1)$ and $y^- = (1, 0)$, then y and y^- are unique nontrivial Boolean elements.

Then τ_h is a state-morphism operator on $\mathbf{M}$ and $(\mathbf{M}, \tau_h)$ is a subdirectly irreducible state-morphism BL-algebra iff $\mathrm{Ker}(h) = \{a \in A \mid h(a) = 1\} = \{1\}$. In such a case, $\mathrm{Ker}(\tau_h) = \{1\} \times B$ and $F := \{1\} \times F_B$ is the least nontrivial state-morphism filter on $\mathbf{M}$.

Now we present the main result on the complete characterization of subdirectly irreducible state-morphism BL-algebras which is a combination of [65, Thm 4.5] and Theorem 5.1.

Theorem 5.2. *A state-morphism BL-algebra* $(\mathbf{M}, \tau)$ *is subdirectly irreducible if and only if one of the following three possibilities holds.*

(i) $\mathbf{M}$ *is linear,* $\tau = \mathrm{Id}_M$ *is the identity on* M, *and the BL-reduct* $\mathbf{M}$ *is a subdirectly irreducible BL-algebra.*

(ii) *The state-morphism operator* τ *is not faithful,* $\mathbf{M}$ *has no nontrivial Boolean elements, and the BL-reduct* $\mathbf{M}$ *of* $(\mathbf{M}, \tau)$ *is a local BL-algebra,* $\mathrm{Ker}(\tau)$ *is a subdirectly irreducible irreducible hoop, and* $\mathrm{Ker}(\tau)$ *and* $\tau(\mathbf{M})$ *have the disjunction property.*

Moreover, $\mathbf{M}$ *is linearly ordered if and only if* $\mathrm{Rad}_1(\mathbf{M})$ *is linearly ordered, and in such a case,* $\mathbf{M}$ *is a subdirectly irreducible BL-algebra such that if* F *is the smallest nontrivial* τ*-filter for* $(\mathbf{M}, \tau)$, *then* F *is the smallest nontrivial BL-filter for* $\mathbf{M}$.

If $\mathrm{Rad}_1(\mathbf{M}) = \mathrm{Ker}(\tau)$, *then* $\mathbf{M}$ *is linearly ordered.*

(iii) *The state-morphism operator* τ *is not faithful,* $\mathbf{M}$ *has a nontrivial Boolean element. There are a linearly ordered BL-algebra* $\mathbf{A}$, *a subdirectly irreducible BL-algebra* $\mathbf{B}$, *and an injective BL-homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *such that* $(\mathbf{M}, \tau)$ *is isomorphic as a state-morphism BL-algebra with the state-morphism BL-algebra* $(\mathbf{A} \times \mathbf{B}, \tau_h)$, *where* $\tau_h(x, y) = (x, h(x))$ *for any* $(x, y) \in A \times B$.

Proof. It follows from [65, Thm 4.5] and Theorem 5.1. □

If t is continuous t-norm we define $x \odot_t y = t(x, y)$ and $x \to_t y = \sup\{z \in [0,1] \mid t(z,x) \leq y\}$ for $x, y \in [0,1]$, then $\mathbb{I}_t := ([0,1]; \min, \max, \odot_t, \to_t, 0, 1)$ is a BL-algebra. Moreover, according to [47, Thm 5.2], the variety of all BL-algebras is generated by all $\mathbb{I}_t$ with a continuous t-norm t. Let $\mathcal{T}$ denote the system of all BL-algebras $\mathbb{I}_t$, where t is any continuous t-norm.

The proof of the following result will follow from Theorem 5.11.

Theorem 5.3. *The variety of all state-morphism BL-algebras is generated by the system* $\{D(\mathbb{I}_t) \mid t \in \mathcal{T}\}$.

5.2 General State-Morphism Algebras

In this section, we generalize the notion of state-morphism BL-algebras to an arbitrary variety of algebras of some type. It is interesting that many results known only for state-morphism MV-algebras or state-morphism BL-algebras have a very general presentation as state-morphism algebras. The main result of this section, Theorem 5.5, says that every subdirectly irreducible state-morphism algebra can be embedded into some diagonal one.

Let $\mathbf{A}$ be any algebra of type F and let $\mathrm{Con}\,\mathbf{A}$ be the lattice of congruences on $\mathbf{A}$ with the least congruence $\Delta_{\mathbf{A}}$. An endomorphisms $\tau : \mathbf{A} \longrightarrow \mathbf{A}$ satisfying $\tau \circ \tau = \tau$ is said to be a *state-morphism* on $\mathbf{A}$ and a couple $(\mathbf{A}, \tau)$ is said to be a *state-morphism algebra* or an algebra with internal state-morphism. Clearly, if $\mathcal{K}$ is a variety of algebras of type F, then the class $\mathcal{K}_\tau$ of all state-morphism algebras $(\mathbf{A}, \tau)$, where $\mathbf{A} \in \mathcal{K}$ and τ is any state-morphism on $\mathbf{A}$, forms a variety, too.

In the rest of the section, we will assume that $\mathbf{A}$ is an arbitrary algebra with a fixed type F; if $\mathbf{A}$ is of a specific type, it will be said that and specified.

Definition 5.1. Let $\mathbf{B} \in \mathcal{K}$. Then an algebra $D(\mathbf{D}) := (\mathbf{B} \times \mathbf{B}, \tau_B)$, where τ_B is defined by $\tau_B(x, y) = (x, x)$, $x, y \in B$, is a state-morphism algebra (more precisely $(\mathbf{B} \times \mathbf{B}, \tau_B) \in \mathcal{K}_\tau$); we call τ_B also a *diagonal state-operator*. If a state-morphism algebra $(\mathbf{C}, \tau)$ can be embedded into some diagonal state-morphism algebra, $(\mathbf{B} \times \mathbf{B}, \tau_B)$, $(\mathbf{C}, \tau)$ is said to be a *subdiagonal* state-morphism algebra*, or, more precisely,* $\mathbf{B}$-*subdiagonal.*

Let $(\mathbf{A}, \tau)$ be a state-morphism algebra. We introduce the following sets:

$$\theta_\tau = \{(x, y) \in A \times A \mid \tau(x) = \tau(y)\}, \tag{3.1}$$

$$\tau(A) = \{\tau(x) \mid x \in A\}.$$

The subalgebra which is an image of $\mathbf{A}$ by τ is denoted by $\tau(\mathbf{A})$ and thus $\tau(\mathbf{A}) \in \mathcal{K}$ and $(\tau(\mathbf{A}), \mathrm{Id}_{\tau(A)}) \in \mathcal{K}_\tau$, where $\mathrm{Id}_{\tau(A)}$ is the identity on $\tau(A)$; we have also $\tau|\tau(A) = \mathrm{Id}_{\tau(A)}$.

If $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we define

$$\theta_\phi := \{(x, y) \in A \times A \mid (\tau(x), \tau(y)) \in \phi\}. \tag{3.2}$$

Finally, if $\phi \subseteq A^2$ then the congruence on $\mathbf{A}$ generated by ϕ is denoted by $\Theta(\phi)$ and the congruence on $(\mathbf{A}, \tau)$ generated by ϕ is denoted by $\Theta_\tau(\phi)$. Clearly $\mathrm{Con}\,(\mathbf{A}, \tau) \subseteq \mathrm{Con}\,\mathbf{A}$ and also $\Theta(\phi) \subseteq \Theta_\tau(\phi)$.

Lemma 5.3. *Let $(\mathbf{A}, \tau)$ be a state-morphism algebra. For any $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we have $\theta_\phi \in \mathrm{Con}\,(\mathbf{A}, \tau)$, and $\theta_\phi \cap \tau(A)^2 = \phi$. In addition, $\theta_\tau \in \mathrm{Con}\,(\mathbf{A}, \tau)$, $\phi \subseteq \theta_\phi$, and $\Theta_\tau(\phi) \subseteq \theta_\phi$.*

Proof. Clearly, θ_ϕ is reflexive and symmetric. Moreover, if $(x, y), (y, z) \in \theta_\phi$, then $\big(\tau(x), \tau(y)\big), \big(\tau(y), \tau(z)\big) \in \phi$ and thus $\big(\tau(x), \tau(z)\big) \in \phi$ which gives $(x, z) \in \theta_\phi$.

Let $f^{\mathbf{A}}$ be an n-ary operation on $\mathbf{A}$ and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ be such that $(x_i, y_i) \in \theta_\phi$ for any $i = 1, \ldots, n$. Then $\big(\tau(x_i), \tau(y_i)\big) \in \phi$ holds for any $i = 1, \ldots, n$. Due to $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we obtain

$$\big(f^{\tau(\mathbf{A})}(\tau(x_1), \ldots, \tau(x_n)), f^{\tau(\mathbf{A})}(\tau(y_1), \ldots, \tau(y_n))\big) \in \phi.$$

Because τ is an endomorphism,

$$\tau(f^{\mathbf{A}}(x_1, \ldots, x_n)) = f^{\tau(\mathbf{A})}(\tau(x_1), \ldots, \tau(x_n))$$

and

$$\tau(f^{\mathbf{A}}(y_1, \ldots, y_n)) = f^{\tau(\mathbf{A})}(\tau(y_1), \ldots, \tau(y_n))$$

which gives

$$\big(\tau(f^{\mathbf{A}}(x_1,\ldots,x_n)),\tau(f^{\mathbf{A}}(y_1,\ldots,y_n))\big)\in\phi$$

and finally also

$$\big(f^{\mathbf{A}}(x_1,\ldots,x_n),f^{\mathbf{A}}(y_1,\ldots,y_n)\big)\in\theta_\phi.$$

Moreover, take an arbitrary $(x,y)\in\theta_\phi$. Then $\big(\tau(\tau(x)),\tau(\tau(y))\big)=\big(\tau(x),\tau(y)\big)\in\phi$ which gives $\big(\tau(x),\tau(y)\big)\in\theta_\phi$.

Hence, $\theta_\phi\in\mathrm{Con}\,(\mathbf{A},\tau)$ and if $\phi=\Delta_{\tau(\mathbf{A})}$, then $\theta_\phi=\theta_\tau$.

It is clear that $\theta_\phi\cap\tau(A)^2\supseteq\phi$. Now let $(x,y)\in\theta_\phi\cap\tau(A)^2$. Then $x,y\in\tau(A)$, $(\tau(x),\tau(y))\in\phi\subseteq\tau(A)^2$, so that $x=\tau(x)\in\tau(A)$, $y=\tau(y)\in\tau(A)$, and consequently, $(x,y)\in\phi$.

It is evident that θ_τ is a congruence on $(\mathbf{A},\tau)$.

Finally, if $(x,y)\in\phi$ then $\tau(x)=x$ and $\tau(y)=y$ which gives

$$\big(\tau(x),\tau(y)\big)=(x,y)\in\phi.$$

Thus $(x,y)\in\theta_\phi$ which finishes the proof that $\phi\subseteq\theta_\phi$ and $\Theta_\tau(\phi)\subseteq\theta_\phi$. $\square$

Lemma 5.4. *Let $\theta\in\mathrm{Con}\,\mathbf{A}$ be such that $\theta\subseteq\theta_\tau$. Then $\theta\in\mathrm{Con}\,(\mathbf{A},\tau)$ holds.*

Moreover, if $x,y\in A$ are such that $(x,y)\in\theta_\tau$, then $\Theta(x,y)=\Theta_\tau(x,y)$.

Proof. If $(x,y)\in\theta\subseteq\theta_\tau$, then $\tau(x)=\tau(y)$ and thus $\big(\tau(x),\tau(y)\big)=\big(\tau(x),\tau(x)\big)\in\theta$ proves that $\theta\in\mathrm{Con}\,(\mathbf{A},\tau)$.

Moreover, if $(x,y)\in\theta_\tau$, then $\Theta(x,y)\subseteq\theta_\tau$. Due to the first part of Lemma, we obtain $\Theta(x,y)\in\mathrm{Con}\,(\mathbf{A},\tau)$ and thus $\Theta_\tau(x,y)\subseteq\Theta(x,y)$ holds. The second inclusion is trivial. $\square$

Lemma 5.5. *If $x,y\in\tau(\mathbf{A})$, then $\Theta(x,y)=\Theta_\tau(x,y)$. Consequently, $\Theta(\phi)=\Theta_\tau(\phi)$ whenever $\phi\subseteq\tau(A)^2$.*

Proof. Let us denote by ϕ the congruence on $\tau(\mathbf{A})$ generated by (x,y). Clearly we obtain the chain of inclusions $\phi\subseteq\Theta(x,y)\subseteq\Theta(\phi)\subseteq\theta_\phi$ (because $(x,y)\in\phi$ and $\phi\subseteq\theta_\phi$, see Lemma 5.3).

Assume $(a, b) \in \Theta(x, y)$, then $(a, b) \in \theta_\phi$ and thus $(\tau(a), \tau(b)) \in \phi \subseteq \Theta(x, y)$. Thus $\Theta(x, y) \in \mathrm{Con}\,(\mathbf{A}, \tau)$ and $\Theta_\tau(x, y) \subseteq \Theta(x, y)$ holds. The second inclusion is trivial.

Finally, let $\phi \subseteq \tau(A)^2$. By [27, Thm 5.3], the both congruence lattices of $\mathbf{A}$ and $(\mathbf{A}, \tau)$ are complete sublattices of the lattice of equivalencies on $\mathbf{A}$, and therefore, they have the same infinite suprema. Hence, by the first part of the lemma,

$$\Theta(\phi) = \bigvee_{(x,y)\in\phi} \Theta(x, y) = \bigvee_{(x,y)\in\phi} \Theta_\tau(x, y) = \Theta_\tau(\phi).$$

□

Remark 5.1. By Lemma 5.3, if ϕ is a congruence on $\tau(\mathbf{A})$, then θ_ϕ is an extension of ϕ on $(\mathbf{A}, \tau)$ and $\Theta(\phi) = \Theta_\tau(\phi) \subseteq \theta_\phi$. There is a natural question whether $\Theta(\phi) = \theta_\phi$? The answer is positive if and only if τ is the identity on A. Indeed, if τ is the identity on A, the statement is evident, in the opposite case, we have $\theta_{\Delta_{\tau(\mathbf{A})}} = \theta_\tau \neq \Delta_{\mathbf{A}} = \Theta(\Delta_{\tau(\mathbf{A})})$.

Theorem 5.4. *Let $(\mathbf{A}, \tau)$ be a subdirectly irreducible state-morphism algebra such that $\mathbf{A}$ is subdirectly reducible. Then there is a subdirectly irreducible algebra $\mathbf{B}$ such that $(\mathbf{A}, \tau)$ is $\mathbf{B}$-subdiagonal.*

Proof. First, if $\theta_\tau = \Delta_{\mathbf{A}}$, then for any $x \in A$, the equality $\tau(x) = x$ holds and thus $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,(\mathbf{A}, \tau)$ which is absurd because $\mathbf{A}$ is subdirectly irreducible and $(\mathbf{A}, \tau)$ is not subdirectly irreducible.

The subdirect irreducibility of $(\mathbf{A}, \tau)$ implies that there is a least proper congruence $\theta_{\min} \in \mathrm{Con}\,(\mathbf{A}, \tau)$. Moreover, due to Lemma 5.4, the congruence $\theta_{\min}$ is also a least proper congruence θ on $\mathbf{A}$ with $\theta \subseteq \theta_\tau$ and thus $\theta_{\min}$ is an atom in $\mathrm{Con}\,\mathbf{A}$. Let us denote

$$\theta_\tau^\perp = \{\theta \in \mathrm{Con}\,\mathbf{A} \mid \theta \cap \theta_\tau = \Delta_{\mathbf{A}}\}.$$

First, we prove that there exists proper $\theta \in \theta_\tau^\perp$. The subdirect reducibility of $\mathbf{A}$ shows that there exists proper $\theta \in \mathrm{Con}\,\mathbf{A}$ with $\theta_{\min} \not\subseteq \theta$. Hence, $\theta_\tau \cap \theta = \Delta_{\mathbf{A}}$ holds (because if $\theta_\tau \cap \theta \neq \Delta_{\mathbf{A}}$, then $\theta_\tau \cap \theta$ is a proper congruence contained in θ_τ and minimality of $\theta_{\min}$ yields $\theta_{\min} \subseteq \theta \cap \theta_\tau \subseteq \theta$, which is absurd).

Moreover, let us have $\theta_n \in \theta_\tau^\perp$ for any $n \in \mathbb{N}$ with $\theta_n \subseteq \theta_{n+1}$, then clearly $\bigvee_{n\in\mathbb{N}} \theta_n = \bigcup_{n\in\mathbb{N}} \theta_n \in \theta_\tau^\perp$. Due to Zorn's Lemma, there is maximal $\theta^* \in \theta_\tau^\perp$.

We have proved that both θ_τ and θ^* are proper congruences on $\mathbf{A}$ with $\theta_\tau \cap \theta^* = \Delta_{\mathbf{A}}$. By the Birkhoff Theorem about subdirect reducibility, $\mathbf{A}$ is a subdirect product of two algebras $\mathbf{A}/\theta_\tau$ and $\mathbf{A}/\theta^*$ with an embedding $h : \mathbf{A} \longrightarrow \mathbf{A}/\theta_\tau \times \mathbf{A}/\theta^*$ defined by $h(x) = (x/\theta_\tau, x/\theta^*)$.

Now we define the mapping $\psi : A/\theta_\tau \longrightarrow A/\theta^*$ by $\psi(x/\theta_\tau) = \tau(x)/\theta^*$. Clearly ψ is well-defined because $x/\theta_\tau = y/\theta_\tau$ yields $\tau(x) = \tau(y)$ and thus $\psi(x/\theta_\tau) = \tau(x)/\theta^* = \tau(y)/\theta^* = \psi(y/\theta_\tau)$. Let us suppose that $\psi(x/\theta_\tau) = \psi(y/\theta_\tau)$. Then $\tau(x)/\theta^* = \tau(y)/\theta^*$ and $\big(\tau(x), \tau(y)\big) \in \theta^*$. Hence, we obtain $\Theta(\tau(x), \tau(y)) \subseteq \theta^*$ holds. Finally, if $\tau(x) \neq \tau(y)$ (thus $\Theta(\tau(x), \tau(y))$ is a proper congruence), then $\tau(x), \tau(y) \in \tau(\mathbf{A})$ and Lemma 5.5 yields $\Theta(\tau(x), \tau(y)) \in \mathrm{Con}\,(\mathbf{A}, \tau)$ and thus $\theta_{\min} \subseteq \Theta(\tau(x), \tau(y)) \subseteq \theta^*$ which is absurd ($\theta_{\min} \subseteq \theta_\tau \cap \theta^* = \Delta_{\mathbf{A}}$). Therefore, the mapping ψ is injective.

We shall prove that ψ is a homomorphism (and thus an embedding). If $f^{\mathbf{A}}$ is an n-ary operation and $x_1/\theta_\tau, \ldots, x_n/\theta_\tau \in \mathbf{A}/\theta_\tau$, then

$$\begin{aligned}
\psi(f^{\mathbf{A}/\theta_\tau}(x_1/\theta_\tau, \ldots, x_n/\theta_\tau)) &= \psi(f^{\mathbf{A}}(x_1, \ldots, x_n)/\theta_\tau) \\
&= \tau(f^{\mathbf{A}}(x_1, \ldots, x_n))/\theta^* \\
&= f^{\mathbf{A}}(\tau(x_1), \ldots, \tau(x_n))/\theta^* \\
&= f^{\mathbf{A}/\theta^*}(\tau(x_1)/\theta^*, \ldots, \tau(x_n)/\theta^*) \\
&= f^{\mathbf{A}/\theta^*}(\psi(x_1/\theta_\tau), \ldots, \psi(x_n/\theta_\tau)).
\end{aligned}$$

Now we prove that $\mathbf{A}$ is $\mathbf{A}/\theta^*$-diagonal. Let $g : A \longrightarrow (A/\theta^*)^2$ be defined via $g(x) = (\psi(x/\theta_\tau), x/\theta^*) = (\tau(x)/\theta^*, x/\theta^*)$. Because the mapping g is the composition of two functions h and ψ which are embeddings, g is also an embedding of $\mathbf{A}$ into $(\mathbf{A}/\theta^*)^2$. Now we can compute:

$$\begin{aligned} g(\tau(x)) &= \big(\tau(\tau(x))/\theta^*, \tau(x)/\theta^*\big) \\ &= \big(\tau(x)/\theta^*, \tau(x)/\theta^*\big) \\ &= \tau_{\mathbf{A}/\theta^*}(\tau(x)/\theta^*, x/\theta^*) \\ &= \tau_{\mathbf{A}/\theta^*}(g(x)), \end{aligned}$$

where $\tau_{\mathbf{A}/\theta^*}$ is the diagonal state-morphism on the product $\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*$. Therefore, $g : (\mathbf{A}, \tau) \longrightarrow (\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*, \tau_{\mathbf{A}/\theta^*})$ is an embedding and $(\mathbf{A}, \tau)$ is $\mathbf{A}/\theta^*$-diagonal.

Finally, we prove the subdirect irreducibility of $\mathbf{A}/\theta^*$. Of course, $\theta_{\min} \cap \theta^* = \Delta_{\mathbf{A}}$ yields $\theta_{\min} \not\subseteq \theta^*$ and thus $\theta^* \subset \theta^* \vee \theta_{\min}$. Moreover, if $\theta^* \subset \theta$, from maximality of θ^* we obtain $\theta \cap \theta_\tau \neq \Delta_{\mathbf{A}}$ and thus $\theta_{\min} \subseteq \theta_\tau \cap \theta$. Finally, $\theta_{\min} \vee \theta^* \subseteq (\theta_\tau \cap \theta) \vee \theta^* \subseteq (\theta_\tau \cap \theta) \vee \theta = \theta$ holds. Hence, for any congruence $\theta \in \operatorname{Con} \mathbf{A}$, the inequality $\theta^* \subset \theta^* \cap \theta_{\min} \subseteq \theta$ holds. Due to the Birkhoff's Theorem and the Second Homomorphism Theorem, an algebra $\mathbf{A}/\theta^*$ is subdirectly irreducible. $\square$

Theorem 5.4 can be extended as follows.

Theorem 5.5. *For every subdirectly irreducible state-morphism algebra* $(\mathbf{A}, \tau)$, *there is a subdirectly irreducible algebra* $\mathbf{B}$ *such that* $(\mathbf{A}, \tau)$ *is* $\mathbf{B}$-*subdiagonal.*

Proof. There are two cases: (1) $(\mathbf{A}, \tau)$ and $\mathbf{A}$ are subdirectly irreducible, and (2) $(\mathbf{A}, \tau)$ is a subdirectly irreducible state-morphism algebra and $\mathbf{A}$ is a subdirectly reducible algebra

(1) Assume that $(\mathbf{A}, \tau)$ and $\mathbf{A}$ are subdirectly irreducible. Define two state-morphism algebras $(\tau(\mathbf{A}) \times \mathbf{A}, \tau_1)$ and $(\mathbf{A} \times \mathbf{A}, \tau_2)$, where $\tau_1(a, b) = (a, a)$, $(a, b) \in \tau(A) \times A$, and $\tau_2(a, b) = (a, a)$, $a, b \in A$. Then the first one is a subalgebra of the second one.

Define a mapping $\phi : A \to \tau(A) \times A$ defined by $\phi(a) = (\tau(a), a)$, $a \in A$. Then ϕ is injective because if $\phi(a) = \phi(b)$ then $(\tau(a), a) = (\tau(b), b)$ and $a = b$. We show that ϕ is a homomorphism. Let $f^{\mathbf{A}}$ be an n-ary operation on $\mathbf{A}$ and let $a_1, \ldots, a_n \in A$. Then

$$\begin{aligned}
\phi(f^{\mathbf{A}}(a_1, \ldots, a_n)) &= \big(\tau(f^{\mathbf{A}}(a_1, \ldots, a_n)), f^{\mathbf{A}}(a_1, \ldots, a_n)\big) \\
&= \big(f^{\mathbf{A}}(\tau(a_1), \ldots, \tau(a_n)), f^{\mathbf{A}}(a_1, \ldots, a_n)\big) \\
&= f^{\tau(\mathbf{A})\times\mathbf{A}}\big((\tau(a_1), a_1), \ldots, (\tau(a_n), a_n)\big) \\
&= f^{\tau(\mathbf{A})\times\mathbf{A}}(\phi(a_1), \ldots, \phi(a_n)).
\end{aligned}$$

Since $\phi : \mathbf{A} \to \tau(\mathbf{A}) \times \mathbf{A} \subseteq \mathbf{A} \times \mathbf{A}$, ϕ can be assumed also as an injective homomorphism from the state-morphism algebra $(\mathbf{A}, \tau)$ into the state-morphism algebra $D(\mathbf{B})$, where $\mathbf{B} := \mathbf{A}$ is a subdirectly irreducible algebra.

(2) This case was proved in Theorem 5.4. $\square$

For example, a state-morphism algebra $(\mathbf{A}, \mathrm{Id}_A)$, where Id_A is the identity on A, is subdirectly irreducible if and only if $\mathbf{A}$ is subdirectly irreducible. Therefore, $(\mathbf{A}, \mathrm{Id}_A)$ can be embedded into $(\mathbf{A} \times \mathbf{A}, \tau_A)$ under the mapping $a \mapsto (a, a)$, $a \in A$. Consequently, every subdirectly irreducible state-morphism algebra $(\mathbf{A}, \mathrm{Id}_A)$ is $\mathbf{A}$-subdiagonal with $\mathbf{A}$ subdirectly irreducible.

We note that in the same way as in [72, Lemma 6.1], it is possible to show that the class of subdiagonal state-morphism algebras is closed under subalgebras and ultraproducts, and not closed under homomorphic images, see [72, Lemma 6.6].

5.3 Varieties of State-Morphism Algebras

In this section, we study varieties of state-morphism algebras and their generators. It is interesting to note that some similar results proved for state-morphism MV-algebras in [72] can be obtained in an analogous way also for a general variety of algebras.

Let τ be a state-morphism operator on an algebra $\mathbf{A}$. We set

$$\mathrm{Ker}(\tau) := \{(x, y) \in A \times A \mid \tau(x) = \tau(y)\},$$

the *kernel* of τ. We say that τ is *faithful* if $\mathrm{Ker}(\tau) = \Delta_{\mathbf{A}}$. It is evident that τ is faithful iff $\tau(x) = x$ for each $x \in A$. In addition, τ is faithful iff τ is injective.

For every class $\mathcal{K}$ of same type algebras, we set $\mathsf{D}(\mathcal{K}) = \{D(\mathbf{A}) \mid \mathbf{A} \in \mathcal{K}\}$, where $D(\mathbf{A}) = (\mathbf{A} \times \mathbf{A}, \tau_A)$.

As usual, given a class $\mathcal{K}$ of algebras of the same type, $\mathsf{I}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$ and $\mathsf{P}(\mathcal{K})$ and $\mathsf{P_U}(\mathcal{K})$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from $\mathcal{K}$, respectively. Moreover, $\mathsf{V}(\mathcal{K})$ will denote the variety generated by $\mathcal{K}$.

Lemma 5.6. (1) *Let $\mathcal{K}$ be a class of algebras of the same type F. Then $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$.*
(2) *Let $\mathcal{V}$ be any variety. Then $\mathcal{V}_\tau = \mathsf{ISD}(\mathcal{V})$.*

Proof. (1) If $D(\mathbf{A}) \in \mathsf{D}(\mathcal{K})$ (thus $\mathbf{A} \in \mathcal{K}$), then the F-reduct of the algebra $D(\mathbf{A})$ is the algebra $\mathbf{A} \times \mathbf{A}$ which belongs to the variety $\mathsf{V}(\mathcal{K})$. Due to definition of $\mathsf{V}(\mathcal{K})_\tau$, we obtain also $D(\mathbf{A}) \in \mathsf{V}(\mathcal{K})_\tau$. We have proved that $\mathsf{D}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$. Because $\mathsf{V}(\mathcal{K})_\tau$ is a variety then also $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$

(2) Let $(\mathbf{A}, \tau) \in \mathcal{V}_\tau$. As we have seen in the proof of Theorem 5.5, the map $\phi : a \mapsto (\tau(a), a)$ is an injective homomorphism of $(\mathbf{A}, \tau)$ into $D(\mathbf{A})$. Hence, ϕ is compatible with τ, and $(\mathbf{A}, \tau) \in \mathsf{ISD}(\mathcal{V})$. Conversely, the F-reduct of any algebra in $\mathsf{D}(\mathcal{V})$ is in $\mathcal{V}$, (being a direct product of algebras in $\mathcal{V}$), and hence the F-reduct of any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathsf{IS}(\mathcal{V}) = \mathcal{V}$. Hence, any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathcal{V}_\tau$. □

Lemma 5.7. *Let $\mathcal{K}$ be a class of algebras of the same type F. Then:*
(1) $\mathsf{DH}(\mathcal{K}) \subseteq \mathsf{HD}(\mathcal{K})$.
(2) $\mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K})$.
(3) $\mathsf{DP}(\mathcal{K}) \subseteq \mathsf{IPD}(\mathcal{K})$.
(4) $\mathsf{VD}(\mathcal{K}) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$.

Proof. (1) Let $D(\mathbf{C}) \in \mathsf{DH}(\mathcal{K})$. Then there are $\mathbf{A} \in \mathcal{K}$ and a homomorphism h from $\mathbf{A}$ onto $\mathbf{C}$. Let for all $a, b \in A$, $h^*(a, b) = (h(a), h(b))$. We claim that h^* is a homomorphism from $D(\mathbf{A})$ onto $D(\mathbf{C})$. That h^* is a homomorphism is clear. We verify that h^* is compatible with τ_A. We have $h^*(\tau_A(a,b)) = h^*(a,a) = (h(a),h(a)) = \tau_C(h(a),h(b)) = \tau_C(h^*(a,b))$. Finally, since h is onto, given $(c,d) \in C \times C$, there are $a,b \in A$ such that $h(a) = c$ and $h(b) = d$. Hence, $h^*(a,b) = (c,d)$, h^* is onto, and $D(\mathbf{C}) \in \mathsf{HD}(\mathcal{K})$.

(2) It is trivial.

(3) Let $\mathbf{A} = \prod_{i \in I}(\mathbf{A}_i) \in \mathsf{P}(\mathcal{K})$, where each $\mathbf{A}_i$ is in $\mathcal{K}$. Then the map

$$\Phi : \big((a_i : i \in I), (b_i : i \in I)\big) \mapsto \big((a_i, b_i) : i \in I\big)$$

is an isomorphism from $D(\mathbf{A})$ onto $\prod_{i \in I} D(\mathbf{A}_i)$. Indeed, it is clear that Φ is an F-isomorphism. Moreover, denoting the state-morphism of $\prod_{i \in I} D(\mathbf{A}_i)$ by τ^*, we get:

$$\Phi\big(\tau_A\big((a_i : i \in I), (b_i : i \in I)\big)\big) = \Phi\big((a_i : i \in I), (a_i : i \in I)\big) =$$
$$= \big((a_i, a_i) : i \in I\big) = \big(\tau_{\mathbf{A}_i}(a_i, b_i) : i \in I\big) = \tau^*\Big(\Phi\big((a_i : i \in I), (b_i : i \in I)\big)\Big),$$

and hence Φ is an isomorphism.

(4) By (1), (2) and (3), $\mathsf{DV}(\mathcal{K}) = \mathsf{DHSP}(\mathcal{K}) \subseteq \mathsf{HSPD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$, and hence $\mathsf{ISDV}(\mathcal{K}) \subseteq \mathsf{ISVD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$. Conversely, by Lemma 5.6(1), $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$, and by Lemma 5.6(2), $\mathsf{V}(\mathcal{K})_\tau = \mathsf{ISDV}(\mathcal{K})$. This proves the claim. □

Theorem 5.6. (1) *For every class $\mathcal{K}$ of algebras of the same type F, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.*

(2) *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of same type algebras. Then $\mathsf{V}(D(\mathcal{K}_1)) = \mathsf{V}(D(\mathcal{K}_2))$ if and only if $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$.*

Proof. (1) By Lemma 5.7(4), $\mathsf{VD}(\mathcal{K}) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$. Moreover, by Lemma 5.6(2), $\mathsf{V}(\mathcal{K})_\tau = \mathsf{ISDV}(\mathcal{K})$. Hence, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.

(2) We have $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathcal{K}_1)_\tau$ and $\mathsf{V}(\mathsf{D}(\mathcal{K}_2)) = \mathsf{V}(\mathcal{K}_2)_\tau$. Clearly, $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$ implies $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$, and hence $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathsf{D}(\mathcal{K}_2))$. Conversely, $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathsf{D}(\mathcal{K}_2))$ implies $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$. But any algebra $\mathbf{A} \in \mathsf{V}(\mathcal{K}_1)$ is the F-reduct of a state-morphism algebra in $\mathsf{V}(\mathcal{K}_1)_\tau$, namely of $(\mathbf{A}, \mathrm{Id}_A)$.

It follows that, if $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$, then the classes of F-reducts of $\mathsf{V}(\mathcal{K}_1)_\tau$ and of $\mathsf{V}(\mathcal{K}_2)_\tau$ coincide, and hence $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$. □

As a direct corollary of Theorem 5.6, we have:

Theorem 5.7. *If a system $\mathcal{K}$ of algebras of the same type F generates the whole variety $\mathcal{V}(F)$ of all algebras of type F, then the variety $\mathcal{V}(F)_\tau$ of all state-morphism algebras $(\mathbf{A}, \tau)$, where $\mathbf{A} \in \mathcal{V}(F)$, is generated by the class $\{D(\mathbf{A}) \mid \mathbf{A} \in \mathcal{K}\}$.*

Some applications of the latter theorem for different varieties of algebras will be done in Section 5.

Theorem 5.8. *If $\mathbf{A}$ is a subdirectly irreducible algebra, then any state-morphism algebra $(\mathbf{A}, \tau)$ is subdirectly irreducible.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible algebra and let τ be a state-morphism operator on $\mathbf{A}$. If τ is the identity on A, then $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,(\mathbf{A}, \tau)$ and, consequently, $(\mathbf{A}, \tau)$ is subdirectly irreducible. If τ is not the identity on A, then θ_τ, defined by (3.1), is a nontrivial congruence on $\mathbf{A}$, and thus $\theta_{\min} \subseteq \theta_\tau$, where $\theta_{\min} \in \mathrm{Con}\,\mathbf{A}$ is the least nontrivial congruence. Hence, $\theta_{\min}$ belongs to the set $\mathrm{Con}\,(\mathbf{A}, \tau)$, see Lemma 5.4. Therefore, $\mathrm{Con}\,(\mathbf{A}, \tau) \subseteq \mathrm{Con}\,\mathbf{A}$ yields the subdirect irreducibility of the algebra $(\mathbf{A}, \tau)$, more precisely, $\theta_{\min}$ is also the least proper congruence in $\mathrm{Con}\,(\mathbf{A}, \tau)$. $\square$

We remind the following Mal'cev Theorem, [27, Lemma 3.1].

Theorem 5.9. *Let $\mathbf{A}$ be an algebra and $\phi \subseteq A^2$. Then $(a, b) \in \Theta(\phi)$ if and only if there exist two finite sequences of terms $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$ and pairs $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ with*

$$a = t_1(\overline{x}_1, a_1),\ t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1}) \text{ and } t_n(\overline{x}_n, b_n) = b$$

for some $\overline{x}_1, \ldots, \overline{x}_n \in A$.

We say that an algebra $\mathbf{B}$ has the Congruence Extension Property (CEP for short) if, for any algebra $\mathbf{A}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and for any congruence $\theta \in \mathrm{Con}\,\mathbf{B}$, there is a congruence $\phi \in \mathrm{Con}\,\mathbf{A}$ such that $\theta = (B \times B) \cap \phi$. A variety $\mathcal{K}$ has the CEP if every algebra in $\mathcal{K}$ has the CEP. For example, the variety of MV-algebra, or the variety of BL-algebras or the variety of state-morphism MV-algebras (see [72, Lemma 6.1]) satisfies the CEP.

Theorem 5.10. *A variety $\mathcal{V}_\tau$ satisfy the CEP if and only if $\mathcal{V}$ satisfies the CEP.*

Proof. Let us have a variety $\mathcal{V}$ with the CEP. If $\mathbf{A} \in \mathcal{V}$ is such that $(\mathbf{A}, \tau)$ is an algebra with state-morphism, for any subalgebra $(\mathbf{B}, \tau) \subseteq (\mathbf{A}, \tau)$ and any $\phi \in \mathrm{Con}\,(\mathbf{B}, \tau)$, the condition $\phi = B^2 \cap \Theta(\phi)$ holds.

Now we prove $\Theta(\phi) = \Theta_\tau(\phi)$. To show that, assume $(a, b) \in \Theta(\phi)$. Mal'cev's Theorem shows the existence of finite sequences of terms

$$t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$$

and pairs $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ with

$$a = t_1(\overline{x}_1, a_1),\ t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1}) \text{ and } t_n(\overline{x}_n, b_n) = b$$

for some $\overline{x}_1, \ldots, \overline{x}_n \in A$. Because τ is an endomorphism, we obtain also equalities

$$\tau(a) = t_1(\tau(\overline{x}_1), \tau(a_1)),\ t_i(\tau(\overline{x}_i), \tau(b_i)) = t_{i+1}(\tau(\overline{x}_{i+1}), \tau(a_{i+1}))$$

and

$$t_n(\tau(\overline{x}_n), \tau(b_n)) = \tau(b).$$

We have assumed that $\phi \in \mathrm{Con}\,(\mathbf{B}, \tau)$, thus $(a_i, b_i) \in \phi$ yields $(\tau(a_i), \tau(b_i)) \in \phi$ for any $i = 1, \ldots, n$. Now, we have obtained $(\tau(a), \tau(b)) \in \Theta(\phi)$. In other words, $\Theta(\phi) \in \mathrm{Con}\,(\mathbf{A}, \tau)$ and thus $\Theta(\phi) = \Theta_\tau(\phi)$.

If $\mathcal{V}_\tau$ has the CEP, then for any $\mathbf{A} \in \mathcal{V}$, we have $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,(\mathbf{A}, \mathrm{Id}_A)$. Clearly, the CEP on $(\mathbf{A}, \mathrm{Id}_A)$ yields the CEP on $\mathbf{A}$. $\square$

5.4 Applications to Special Types of Algebras

In this section, we apply a general result concerning generators of some varieties of state-morphism algebras, Theorem 5.6, to the variety of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MV-algebras, when we use different systems of t-norms on the real interval $[0, 1]$ and a special type of pseudo MV-algebras, respectively.

Algebras for which the logic MTL is sound are called MTL-algebras. They can be characterized as prelinear commutative bounded integral residuated lattices. In more detail, according to [77], an algebraic structure $\mathbf{A} = (A; \wedge, \vee, *, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ is an *MTL-algebra* if it satisfies conditions (BL1), (BL2), (BL3) and (BL5) from Definition 1.4.

If t is any continuous t-norm on $[0, 1]$, we define two binary operations $*_t \to_t$ on $[0, 1]$ via $x *_t y = t(x, y)$ and $x \to_t y = \sup\{z \in [0, 1] \mid t(z, x) \leq y\}$ for $x, y \in [0, 1]$, then $\mathbb{I}_t = ([0, 1]; \min, \max, *_t, \to_t, 0, 1)$ is an example of an MTL-algebra. An MTL-algebra $\mathbb{I}_t$ is a BL-algebra iff t is continuous.

Due to [77], the class $\mathcal{T}_{lc}$, which denotes the system of all BL-algebras $\mathbb{I}_t$, where t is a continuous t-norm on the interval $[0, 1]$, generates the variety of MTL-algebras. This result was strengthened in [144] who introduced the class of regular continuous t-norms which is strictly smaller than the class of continuous t-norms, but they generate the variety of MTL-algebras. Finally, we recall that a noncommutative generalization of MV-algebras was introduced in [88] as *pseudo MV-algebras* or in [133] as *generalized MV-algebras*. According to [65], every pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$ is an interval in a unital ℓ-group (G, u) with strong unit u, i.e. $M \cong \Gamma(G, u) := [0, u]$, where $x \oplus y = (x+y)\wedge$, $x^- = u - x$, $x^\sim = -x + u$, $0 = 0$, and $1 = u$. If (G, u) is double transitive (for definitions and details see [71]), then $\Gamma(G, u)$ generates the variety of pseudo MV-algebras, [71, Thm 4.8]. For example, if $\mathrm{Aut}(\mathbb{R})$ is the set of all automorphisms of the real line $\mathbb{R}$ preserving the natural order in $\mathbb{R}$ and $u(t) := t + 1$, $t \in \mathbb{R}$, let $\mathrm{Aut}_u(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R}) \mid g \leq nu$ for some integer $n \geq 1\}$. Then $\Gamma(\mathrm{Aut}_u(\mathbb{R}), u)$ is double transitive and it generates the variety of pseudo MV-algebras, see [71, Ex 5.3].

Now we apply the general statement, Theorem 5.7, on generators to different types of state-morphism algebras. We recall that $\mathcal{T}$ was defined as the class of all BL-algebras $\mathbb{I}_t$, where t is a continuous t-norm on $[0, 1]$.

Theorem 5.11. (1) *The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0, 1]_{MV})$.*

(2) *The variety of all state-morphism BL-algebras is generated by the class* $\{D(\mathbb{I}_t) \mid \mathbb{I}_t \in \mathcal{T}\}$.

(3) *The variety of all state-morphism MTL-algebras is generated by the class* $\{D(\mathbb{I}_t) \mid \mathbb{I}_t \in \mathcal{T}_{lc}\}$.

(4) *The variety of all state-morphism naBL-algebras is generated by the class* $\{D(\mathbb{I}_t^{na}) \mid \mathbb{I}_t \in na\mathcal{T}\}$.

(5) *If a unital* ℓ*-group* (G, u) *is double transitive, then* $D(\Gamma(G, u))$ *generates the variety of state-morphism pseudo MV-algebras.*

Proof. (1) It follows from the fact that the MV-algebra of the real interval $[0, 1]$ generates the variety of MV-algebras, see e.g. [46, Prop 8.1.1], and then apply Theorem 5.7.

(2) The statement follows from the fact that $\mathsf{V}(\mathcal{T})$ is by [47, Thm 5.2] the variety $\mathcal{BL}$ of all BL-algebras. Now it suffices to apply Theorem 5.7.

(3) By [77], the class $\mathcal{T}_{lc}$ of all $\mathbb{I}_t$, where t is any left-continuous t-norms on the interval $[0, 1]$, generates the variety of MTL-algebras; then apply Theorem 5.7.

(4) By [19, Thm 8], the class $na\mathcal{T}$ of all $\mathbb{I}_t$, where t is any non-associative t-norms on the interval $[0, 1]$, generates the variety of non-associative BL-algebras; then apply again Theorem 5.7.

(5) By the above, $\Gamma(G, u)$ generates the variety of pseudo MV-algebras, see also [71, Thm 4.8]; then apply Theorem 5.7. $\square$

We note that the case (1) in Theorem 5.7 was an open problem posed in [53] and was positively solved in [72, Thm 5.4(3)].

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