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Approximating closest homomorphisms into Boolean CSP-templates

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Motivation: coding theory

(Block) code…

…a *finitary relation* $C \subseteq A^n$ on a finite set, called alphabet.

Remark

Often: $A = \{0, 1\}$ and $C \subseteq \{0, 1\}^n$ linear subspace of $\mathrm{GF}(2)^n$;
C… linear code

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Nearest neighbour decoding

- message $\leftrightarrow M \subseteq C \subseteq A^n \overset{\text{noise}}{\rightsquigarrow} M' \subseteq A^n \overset{?}{\to} M \leftrightarrow$ message
- $M \ni m \overset{\text{noise}}{\mapsto} m' \in A^n \mapsto m'' \overset{?}{=} m$
- $m'' := \operatorname{argmin}_{c \in C} \mathrm{d_H}(c, m')$ (a codeword closest to m')

Which "closest"?—Hamming distance

Hamming distance

$$\begin{aligned} \mathrm{d_H}\colon \quad A^n \times A^n &\longrightarrow \{0,\dots,n\} \subseteq \mathbb{R}_{\geq 0} \\ (m, m') &\longmapsto \mathrm{d_H}(m, m') := |\{\, i \in n \mid m_i \neq m'_i \}| \end{aligned}$$

(count places where strings differ)

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$(A^n, \mathrm{d_H})\dots$ metric space for all $n \in \mathbb{N}$.

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$(A^n, \mathrm{d_H})$ ... metric space for all $n \in \mathbb{N}$.

Toy example

- $A = \{1, 9, \mathrm{n}, \mathrm{o}, \mathrm{r}, \ddot{\mathrm{u}}, \mathrm{A}, \mathrm{B}\}$, $n = 9$
- $m = (\mathrm{A}, \mathrm{A}, \mathrm{A}, 9, 1, \mathrm{B}, \mathrm{r}, \mathrm{n}, \mathrm{o})$

  $m' = (\mathrm{A}, \mathrm{A}, 1, 9, \mathrm{B}, \mathrm{r}, \ddot{\mathrm{u}}, \mathrm{n}, \mathrm{n})$

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  $m' = (\mathrm{A}, \mathrm{A}, 1, 9, \mathrm{B}, \mathrm{r}, \ddot{\mathrm{u}}, \mathrm{n}, \mathrm{n})$
- $\mathrm{d_H}(m, m') = 5$

Our problem

General assumption

Everything will be Boolean from now on! $A = \{0, 1\}$

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We consider...

- codes (Boolean relations) $C \subseteq \{0,1\}^n$
  $\equiv$ solutions (models) of conjunctive formulæ φ
  over a Boolean constraint language Γ
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We want...

- $m' \in C \setminus \{m\}$ (feasibility)
- $d_H(m, m') = \min \{ d_H(m, c) \mid c \in C \setminus \{m\} \}$ (optimality)

Nearest Other Solution—more formally. . .

Boolean constraint language

. . . a finite set Γ of finitary relations on $\{0, 1\}$.

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Problem NOSol(Γ)

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Input:
- $\varphi = \bigwedge_{i \in I} R_i(\boldsymbol{v}_i)$ (I finite, $R_i \in \Gamma$)
  over variables from a **minimal** set V
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- $m \in C_\varphi$

Associated "code": $C_\varphi = \left\{ m' \in \{0,1\}^V \;\middle|\; m' \models \varphi \right\}$

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Optimisation objective: $\mathrm{d_H}(m', m) \longrightarrow \min$

Concrete example

NOSol

- $R_1 := \left\{ (x_1, x_2, x_3) \in \{0,1\}^3 \;\middle|\; x_1 \oplus x_2 \oplus x_3 = 1 \right\}$
- $R_2 := \left\{ (x_1, x_2) \in \{0,1\}^2 \;\middle|\; x_1 \leq x_2 \right\}$
- $\Gamma := \{R_1, R_2\}$
- $\varphi = R_1(x_1, x_2, x_3) \wedge R_1(x_2, x_3, x_3) \wedge R_2(x_3, x_4) \wedge R_2(x_1, x_5)$

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- $\min \left\{ \mathrm{d_H}(m, m') \mid m' \in C_\varphi \setminus \{m\} \right\} = 1.$

Even more formally...

Γ ... Boolean constraint language, V a finite set (of variables).

Constraint over Γ and V

... a pair $(R, (v_1, \ldots, v_n))$ s.t. $R \in \Gamma$, n-ary, $(v_1, \ldots, v_n) \in V^n$

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Satisfaction relation (assignment $s \in \{0,1\}^V$)

$s\colon V \longrightarrow T \models (R, (v_1, \ldots, v_n)) \iff s \circ (v_1, \ldots, v_n) \in R.$

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Distances between homomorphisms. . .

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NOSol(Γ) ... differently

Instance: a finite relational structure $\underset{\sim}{\mathbf{V}} = \left\langle V; \left(\tilde{R}\right)_{R\in\Gamma} \right\rangle$

$$V = \bigcup \left\{ \{v_1, \ldots, v_n\} \;\middle|\; (v_1, \ldots, v_n) \in \tilde{R}, R \in \Gamma \right\}$$

$m \in \hom\left(\underset{\sim}{\mathbf{V}}, \underset{\sim}{\mathbf{T}}\right)$.

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Optimisation criterion: $\mathrm{d_H}(m, m') \longrightarrow \min!$

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$$\varphi = \bigwedge_{i \in I} R_i(\boldsymbol{v}_i) \rightsquigarrow \left(\tilde{R} := \{ \boldsymbol{v}_i \mid \exists i \in I \colon R_i = R \}\right)_{R \in \Gamma}$$

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- prove that $\mathsf{NOSol}_{\exists,=}(\Gamma) \leq_{\mathsf{AP}} \mathsf{NOSol}(\Gamma)$

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- prove that $\mathsf{NOSol}_{\exists,=}(\Gamma) \leq_{\mathsf{AP}} \mathsf{NOSol}(\Gamma)$
- conclude $\Gamma' \subseteq [\Gamma]_{\mathsf{R}_2} \implies \mathsf{NOSol}(\Gamma') \leq_{\mathsf{AP}} \mathsf{NOSol}(\Gamma)$

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- conclude $\Gamma' \subseteq [\Gamma]_{\mathsf{R}_2} \implies \mathsf{NOSol}(\Gamma') \leq_{\mathsf{AP}} \mathsf{NOSol}(\Gamma)$
- use clone theory and Post's lattice to classify the complexity

Keep dreaming!

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I am not going to do that, since. . .

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- I don't have time for that
- **It simply does not work like that.**

Demotivating example

NOSol $(\{R\})$

- $R = \left\{ m_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, m_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, m_2 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
- $C_{R(x_1,x_2,x_3)} = R$, m_0, optimal solution m_1

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  no feasible solutions

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- $C_{\exists x_2 (R(x_1,x_2,x_3))}$, $m_0' = m_1' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

  optimal (and only feasible) solution $m_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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- $C_{\exists x_2 (R(x_1,x_2,x_3))}$, $m'_0 = m'_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
  optimal (and only feasible) solution $m'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- witness and optimal value can change

Conjunctive closure only!

Operational side	Closure of relations under . . .
total operations: clones (Bodnarčuk, Kalužnin, Kotov, Romov 1969)	$\exists, \wedge, =$
partial operations: strong partial clones (Romov 1981)	$\wedge, =$
hyperoperations: hyperclones (Kotov, Romov 1970(?), Rosenberg 1996, Romov 1997)	$\exists, \wedge$
partial hyperoperations : strong partial hyperclones (Romov 2002, 2006)	$\wedge$

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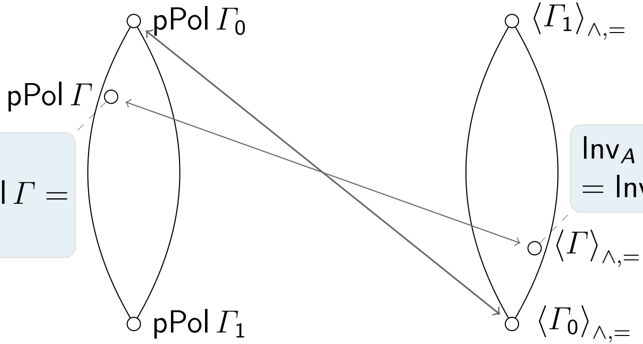
partial operations: strong partial clones (Romov 1981)	$\wedge, =$ ☯ weak systems
hyperoperations: hyperclones (Kotov, Romov 1970(?), Rosenberg 1996, Romov 1997)	$\exists, \wedge$ ☹
partial hyperoperations ☹: strong partial hyperclones (Romov 2002, 2006)	$\wedge$ ☺

Strong partial clones & weak bases

interval of strong partial clones covering $C \leq \mathrm{O}_A$

generating systems of $\mathrm{Inv}_A\, C$ mod $\langle\ \rangle_{\wedge,=}$



$C = \mathrm{O}_A \cap \mathrm{pPol}\, \Gamma = \mathrm{Pol}_A\, \Gamma$

$\mathrm{Inv}_A\, \mathrm{Pol}_A\, \Gamma = \mathrm{Inv}_A\, C$

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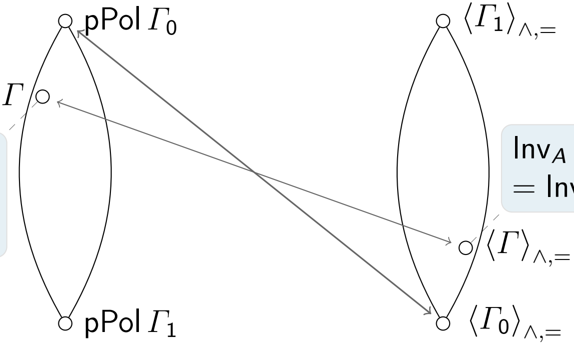


$C = \mathrm{O}_A \cap \mathrm{pPol}\, \Gamma = \mathrm{Pol}_A\, \Gamma$

$\mathrm{Inv}_A\, \mathrm{Pol}_A\, \Gamma = \mathrm{Inv}_A\, C$

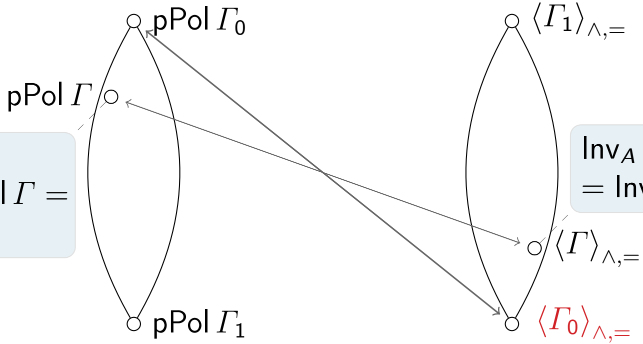
$\mathcal{G} := \{ \Gamma \subseteq \mathrm{R}_A \mid \mathrm{Inv}_A\, \mathrm{Pol}_A\, \Gamma = \mathrm{Inv}_A\, C \}$

$\Gamma \preceq \Gamma' \;:\Longleftrightarrow\; \langle \Gamma \rangle_{\wedge,=} \subseteq \langle \Gamma' \rangle_{\wedge,=}$

Strong partial clones & weak bases

interval of strong partial clones covering $C \leq \mathrm{O}_A$

generating systems of $\mathrm{Inv}_A\, C$ mod $\langle\ \rangle_{\wedge,=}$



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weak base of $\mathrm{Inv}_A\, C$: finite smallest Γ in quasiorder $(\mathcal{G}, \preceq)$

Weak bases & irredundant relations

$\mathcal{G} := \{\, \Gamma \subseteq \mathrm{R}_A \mid \mathrm{Inv}_A \mathrm{Pol}_A \Gamma = Q \}$ $\quad$ $\Gamma \preceq \Gamma' :\Longleftrightarrow \langle \Gamma \rangle_{\wedge,=} \subseteq \langle \Gamma' \rangle_{\wedge,=}$

Weak base of a relational clone Q (Schnoor[2], 2008)

a finite set $\Gamma \subseteq \mathrm{R}_A$, smallest in $(\mathcal{G}, \preceq)$

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$\preceq$-smallest $\Gamma \in \mathcal{G}$ means. . . Corollary 4.3 in (Schnoor[2], 2008):

$\mathrm{Inv}_A \mathrm{Pol}_A \Gamma = Q$ and

$\forall \Gamma' \subseteq \mathrm{R}_A\colon\ \mathrm{Inv}_A \mathrm{Pol}_A \Gamma' = Q \implies \Gamma \subseteq \langle \Gamma' \rangle_{\wedge,=}$

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Irredundant relation $R \subseteq A^n$

R... neither duplicate nor fictitious coordinates.

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For $R \subseteq A^n$ irredundant

$R \in \langle\Gamma\rangle_{\wedge,=} \implies R \in \langle\Gamma\rangle_{\wedge}$

Minimal weak bases

Γ weak base of relational clone Q

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$\Gamma \subseteq \mathrm{R}_A$ weak base of Q, all $R \in \Gamma$ irredundant $\implies$

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Minimal weak base Γ of relational clone Q (Lagerkvist, 2014)

$\Gamma = \{R\}$ with $R \subseteq A^n$ irredundant,
R $\subseteq$-minimal w.r.t. $\mathrm{Inv}_A \mathrm{Pol}_A \{R\} = Q$.

Minimal weak bases

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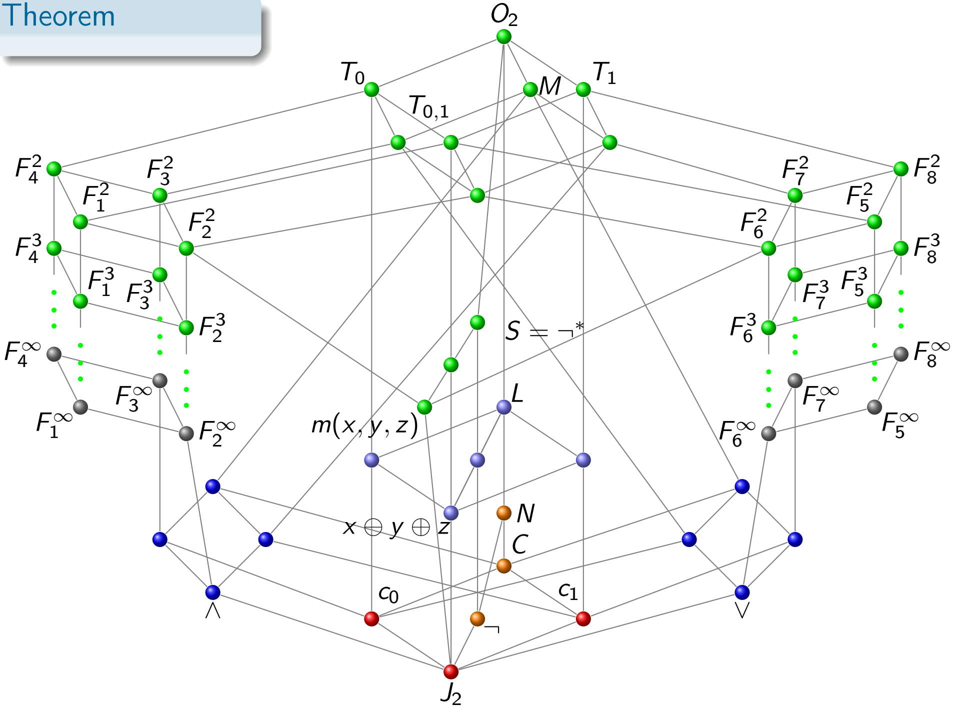
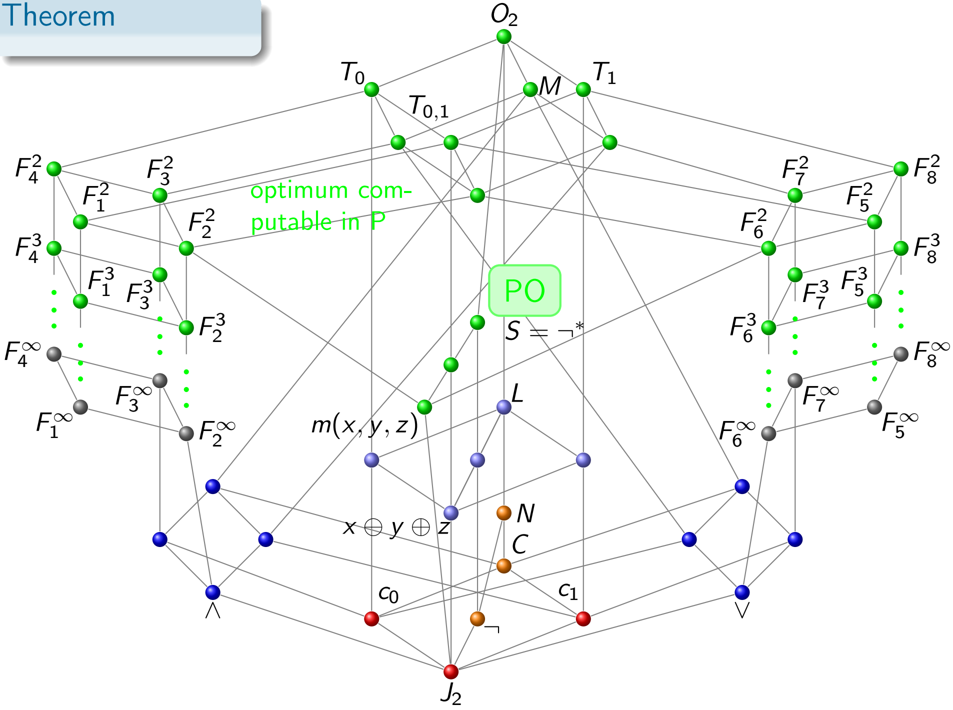
Lagerkvist computed minimal weak bases for all finitely related Boolean clones

Theorem



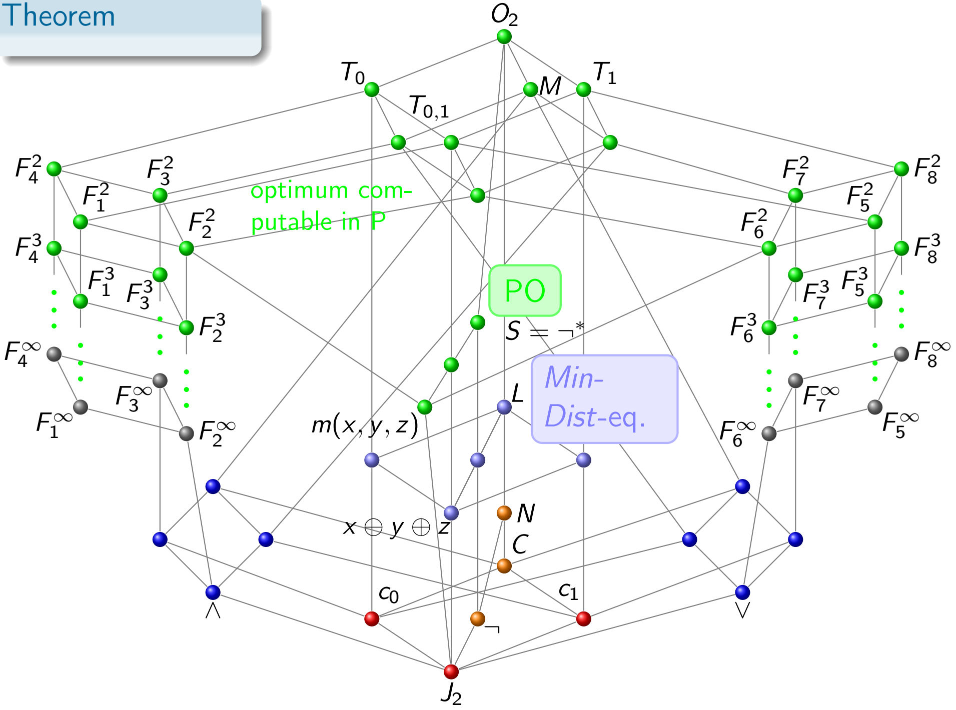
O_2
T_0
M
T_1
$T_{0,1}$
F_4^2
F_3^2
F_1^2
F_2^2
F_7^2
F_8^2
F_5^2
F_6^2
F_4^3
F_1^3
F_3^3
F_2^3
F_8^3
F_5^3
F_7^3
F_6^3
$S = \neg^*$
F_4^∞
F_3^∞
F_1^∞
F_2^∞
F_8^∞
F_7^∞
F_6^∞
F_5^∞
L
$m(x, y, z)$
N
$x \oplus y \oplus z$
C
c_0
c_1
$\wedge$
$\vee$
$\neg$
J_2

Theorem



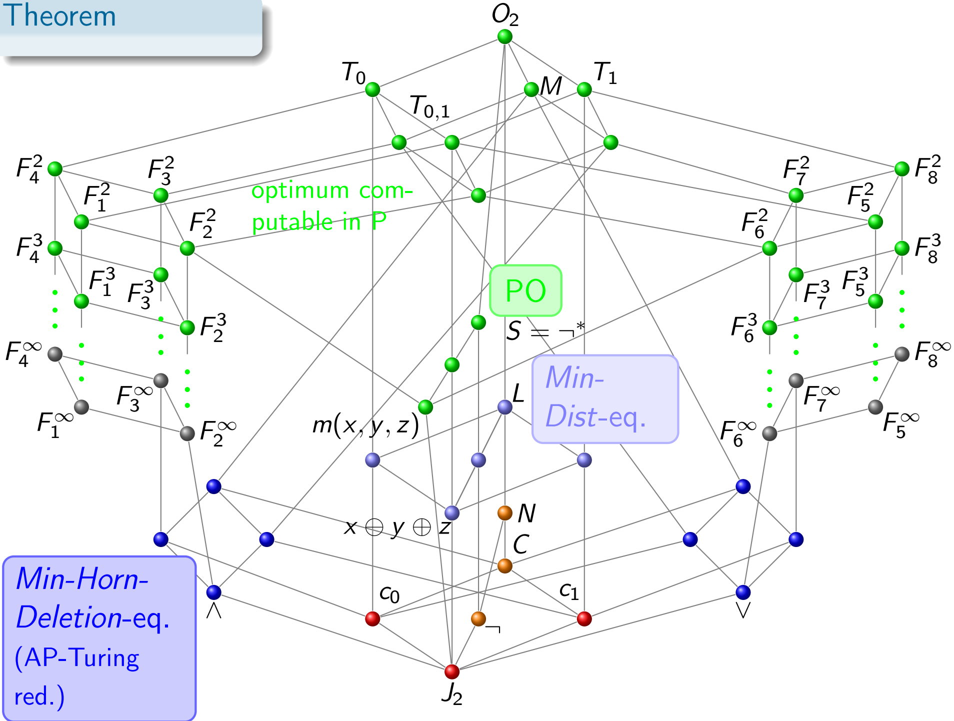
O_2
T_0
T_1
M
$T_{0,1}$
F_4^2
F_3^2
F_1^2
F_2^2
F_7^2
F_8^2
F_5^2
F_6^2
optimum com-
putable in P
F_4^3
F_1^3
F_3^3
F_2^3
F_8^3
F_5^3
F_7^3
F_6^3
PO
$S = \neg^*$
F_4^∞
F_3^∞
F_1^∞
F_2^∞
F_8^∞
F_7^∞
F_6^∞
F_5^∞
L
$m(x, y, z)$
N
$x \oplus y \oplus z$
C
c_0
c_1
$\wedge$
$\vee$
$\neg$
J_2

Theorem



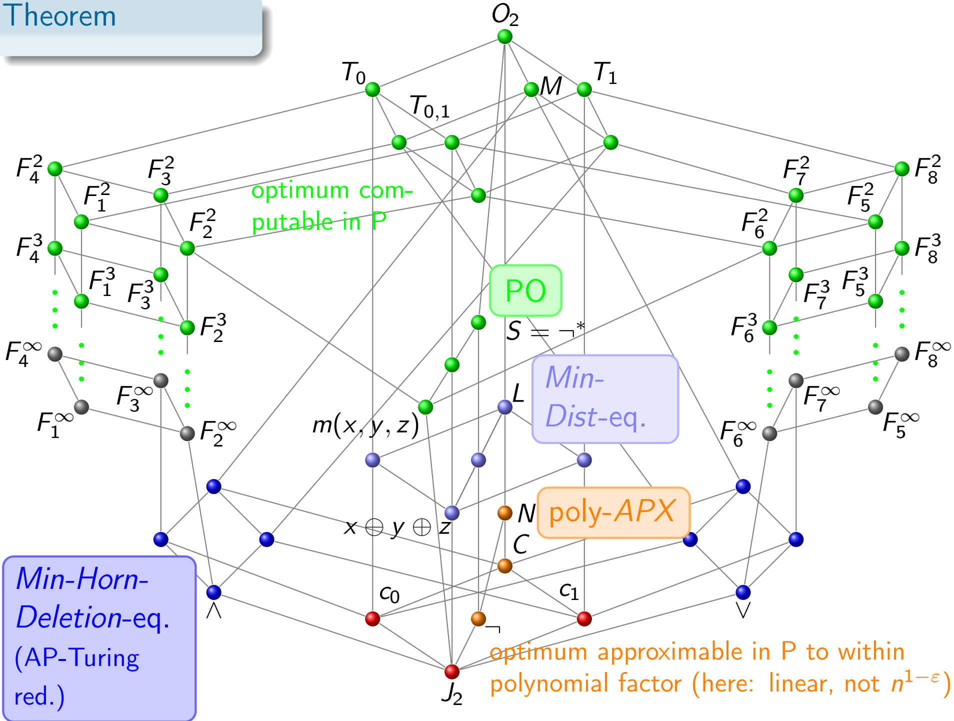
O_2
T_0
M
T_1
$T_{0,1}$
F_4^2
F_3^2
F_1^2
F_2^2
optimum computable in P
F_7^2
F_8^2
F_5^2
F_6^2
F_4^3
F_1^3
F_3^3
F_2^3
F_8^3
F_5^3
F_7^3
F_6^3
PO
$S = \neg^*$
F_4^∞
F_3^∞
F_1^∞
F_2^∞
L
Min-Dist-eq.
F_8^∞
F_7^∞
F_6^∞
F_5^∞
$m(x, y, z)$
$x \oplus y \oplus z$
N
C
c_0
c_1
$\wedge$
$\vee$
$\neg$
J_2

Theorem



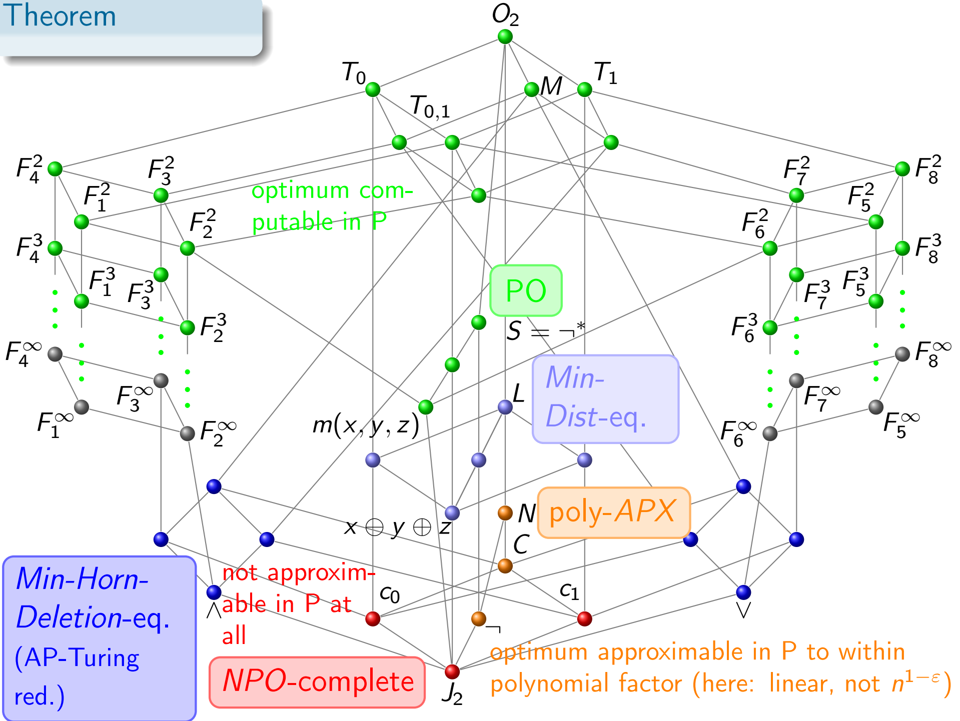
O2
T0
T1
M
T0,1
F4^2
F3^2
F1^2
F2^2
F4^3
F1^3
F3^3
F2^3
F4^∞
F3^∞
F1^∞
F2^∞
F7^2
F8^2
F5^2
F6^2
F8^3
F7^3
F5^3
F6^3
F8^∞
F7^∞
F6^∞
F5^∞
optimum computable in P
PO
S = ¬*
Min-Dist-eq.
L
m(x, y, z)
x ⊕ y ⊕ z
N
C
c0
c1
∧
∨
¬
J2
Min-Horn-Deletion-eq.
(AP-Turing red.)

Theorem



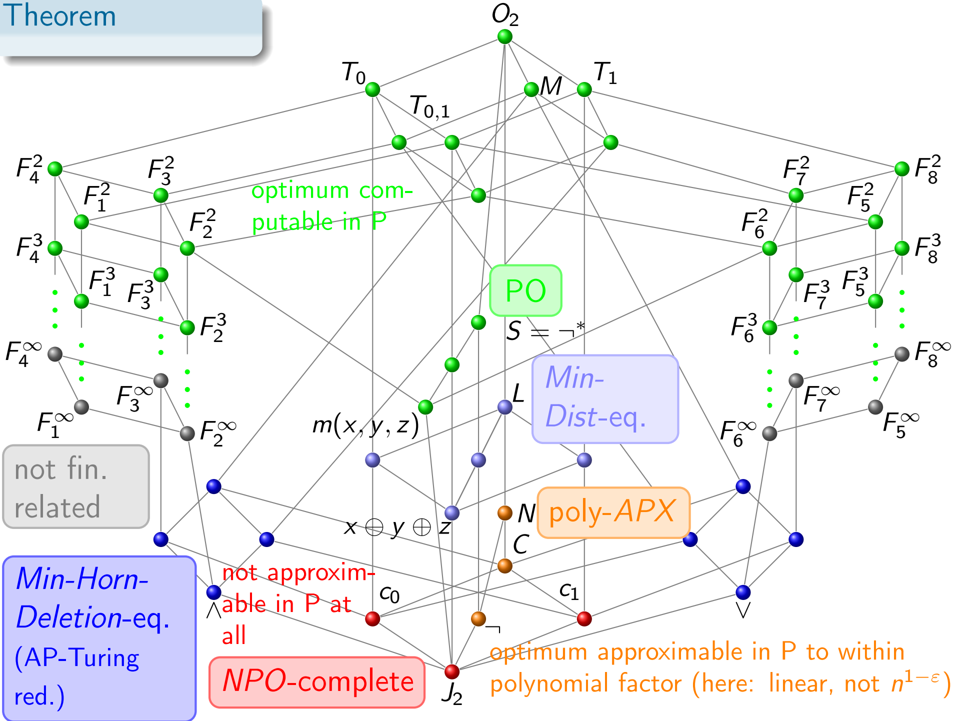
O_2
T_0
T_1
M
$T_{0,1}$
F_4^2
F_3^2
F_1^2
F_2^2
F_4^3
F_1^3
F_3^3
F_2^3
F_4^∞
F_3^∞
F_1^∞
F_2^∞
F_7^2
F_8^2
F_5^2
F_6^2
F_8^3
F_5^3
F_7^3
F_6^3
F_8^∞
F_7^∞
F_6^∞
F_5^∞
optimum com-
putable in P
PO
$S = \neg^*$
Min-
Dist-eq.
L
$m(x, y, z)$
$x \oplus y \oplus z$
N
poly-APX
C
c_0
c_1
$\wedge$
$\vee$
$\neg$
J_2
Min-Horn-
Deletion-eq.
(AP-Turing
red.)
optimum approximable in P to within
polynomial factor (here: linear, not $n^{1-\varepsilon}$)

Theorem



O_2
T_0
T_1
M
$T_{0,1}$
F_4^2
F_3^2
F_1^2
F_2^2
F_7^2
F_8^2
F_5^2
F_6^2
optimum computable in P
F_4^3
F_1^3
F_3^3
F_2^3
F_8^3
F_5^3
F_7^3
F_6^3
PO
$S = \neg^*$
F_4^∞
F_3^∞
F_1^∞
F_2^∞
F_8^∞
F_7^∞
F_6^∞
F_5^∞
L
Min-Dist-eq.
$m(x, y, z)$
$x \oplus y \oplus z$
N
poly-APX
C
Min-Horn-Deletion-eq. (AP-Turing red.)
not approximable in P at all
c_0
c_1
$\wedge$
$\neg$
$\vee$
NPO-complete
J_2
optimum approximable in P to within polynomial factor (here: linear, not $n^{1-\varepsilon}$)

Theorem



O_2
T_0
M
T_1
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F_4^2
F_3^2
F_1^2
F_2^2
F_7^2
F_8^2
F_5^2
F_6^2
optimum computable in P
F_4^3
F_1^3
F_3^3
F_2^3
F_8^3
F_3^3
F_7^3
F_5^3
F_6^3
PO
$S = \neg^*$
F_4^∞
F_3^∞
F_1^∞
F_2^∞
F_8^∞
F_7^∞
F_6^∞
F_5^∞
Min-Dist-eq.
L
$m(x, y, z)$
not fin. related
$x \oplus y \oplus z$
N
poly-APX
C
Min-Horn-Deletion-eq. (AP-Turing red.)
not approximable in P at all
c_0
c_1
$\wedge$
$\vee$
$\neg$
NPO-complete
J_2
optimum approximable in P to within polynomial factor (here: linear, not $n^{1-\varepsilon}$)