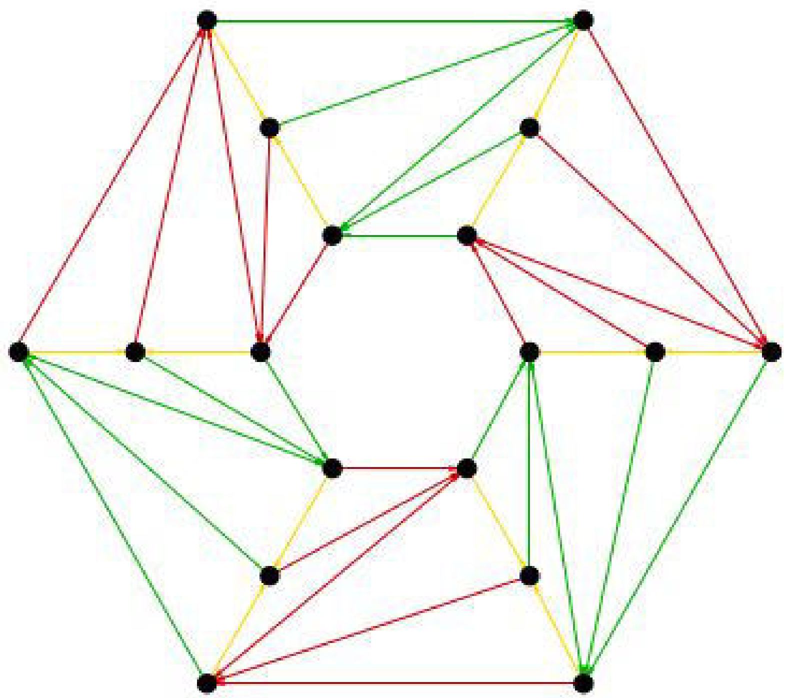
Planar Right Groups

– cayley graphs of semigroups –

Ulrich Knauer

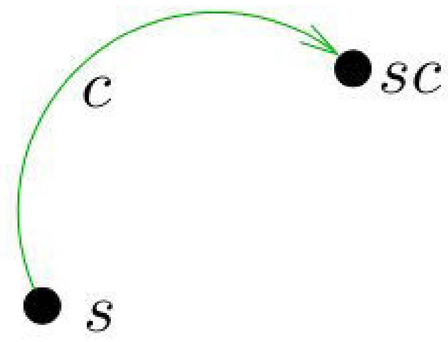
joint work with Kolja Knauer

Brno, 2016



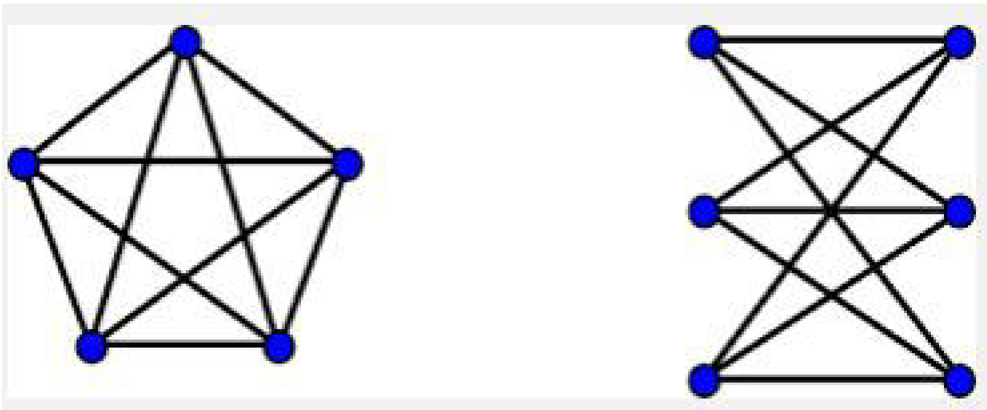
directed (right) cayley graph

S semigroup and $C \subseteq S$ then $\mathrm{Cay}(S, C) := (V, A)$, where $V = S$ and $(s, t) \in A$ iff $sc = t$ for some $c \in C$.



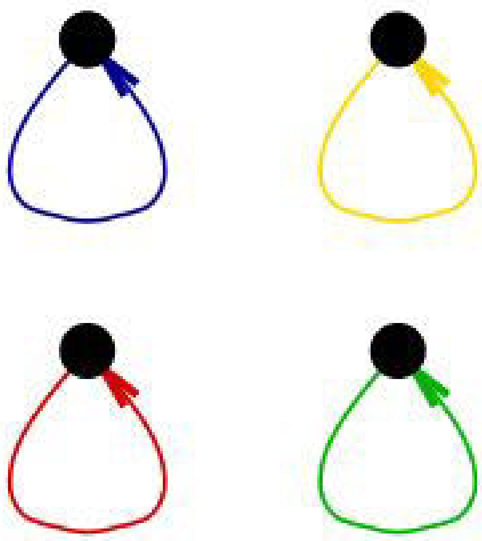
planar group

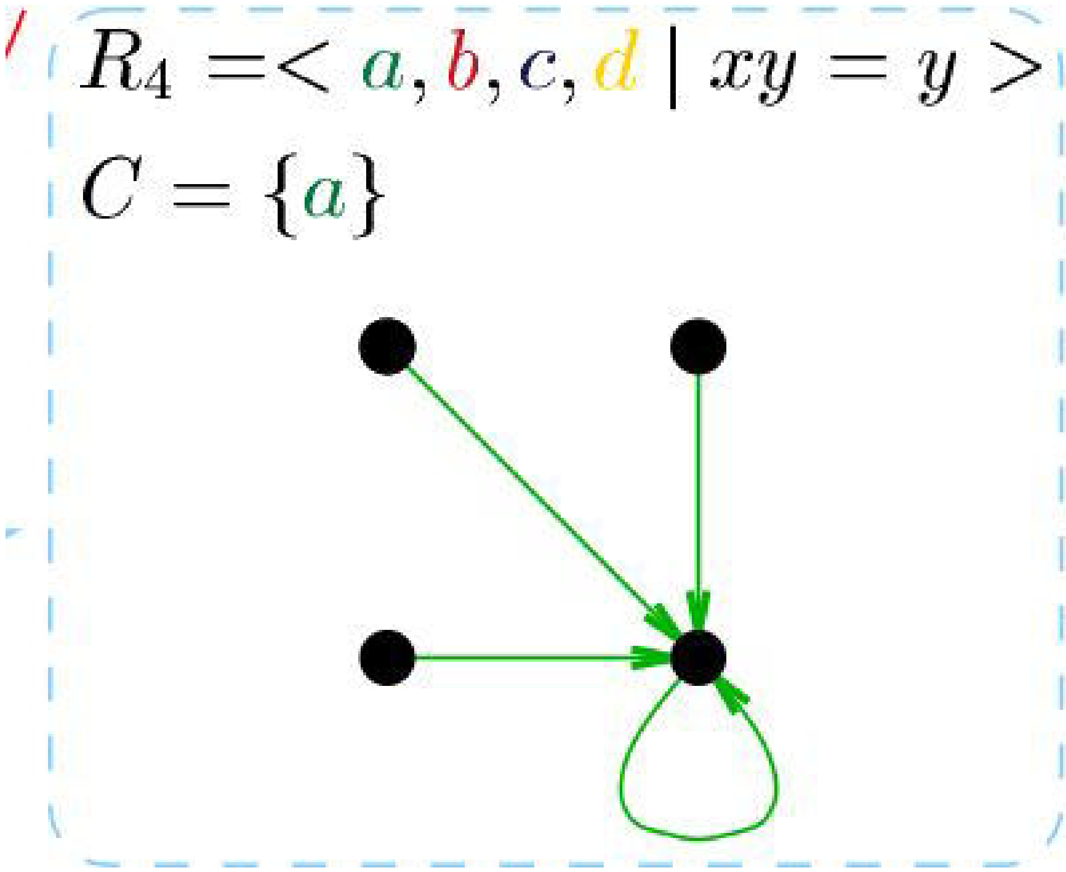
G group planar if there is generating set $C \subseteq G$ such that $\mathrm{Cay}(G, C)$ is planar



$$L_4 = < a, b, c, d \mid xy = x >$$

$$C = \{a, b, c, d\}$$



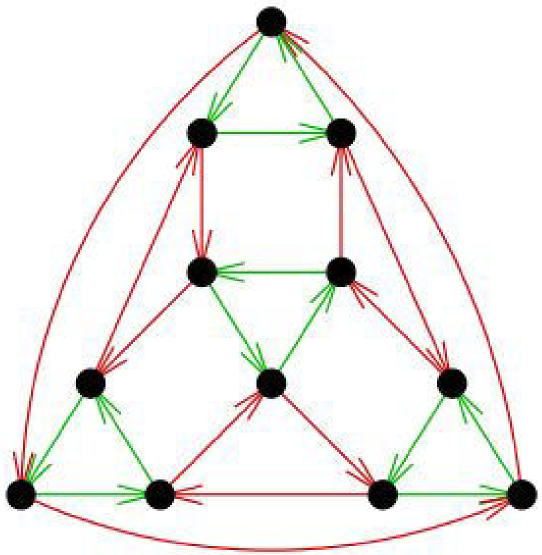

$R_4 =< a, b, c, d \mid xy = y >$
$C = \{a\}$

Maschke's Theorem 1896

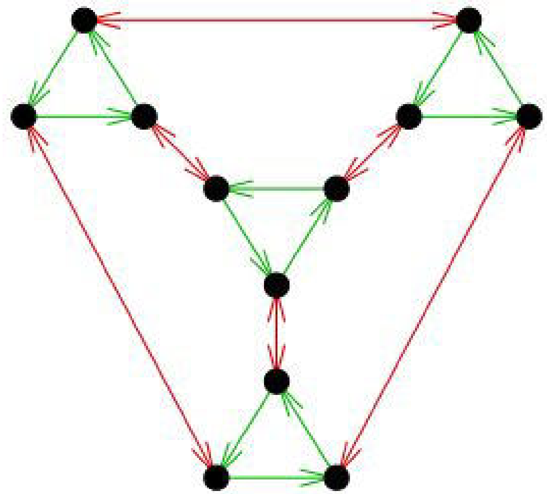
the (finite) planar groups are exactly:

$$\mathbb{Z}_n, D_n, A_4, S_4, A_5, \mathbb{Z}_2 \times \mathbb{Z}_n, \mathbb{Z}_2 \times D_n, \mathbb{Z}_2 \times A_4, \mathbb{Z}_2 \times S_4, \mathbb{Z}_2 \times A_5$$

... the discrete isometry groups of the sphere.



Cay(A_4,{(123),(234)})



Cay(A_4,{(123),(12)(34)})

Planar groups (Maschke 1896)

Z_n

cyclic

D_n, A_4, S_4, A_5, $Z_2 \times A_4$, $Z_2 \times Z_{2n}$

all with 2 generators a, b

and

$Z_2 \times D_{2n}$, $Z_2 \times S_4$, $Z_2 \times A_5$

with 3 generators a, b, c of order 2

Note that

$Z_2 \times Z_{2n+1} = Z_{4n+2}$

$Z_2 \times D_{2n+1} = D_{4n+2}$

$Z_2 \times A_5$ can be generated by two elements, but then the Cayley graph will not be planar.

Maschke's Theorem 1896

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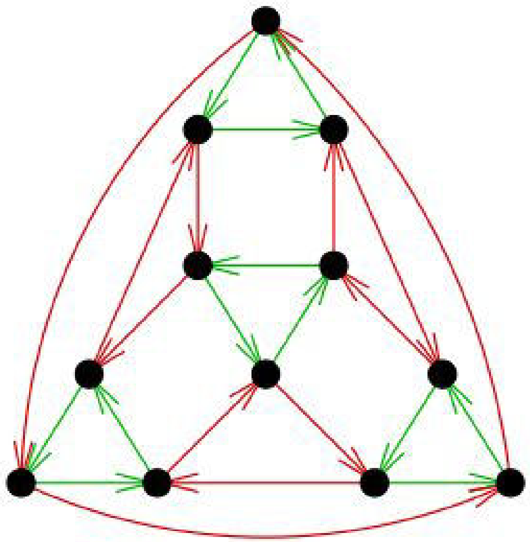
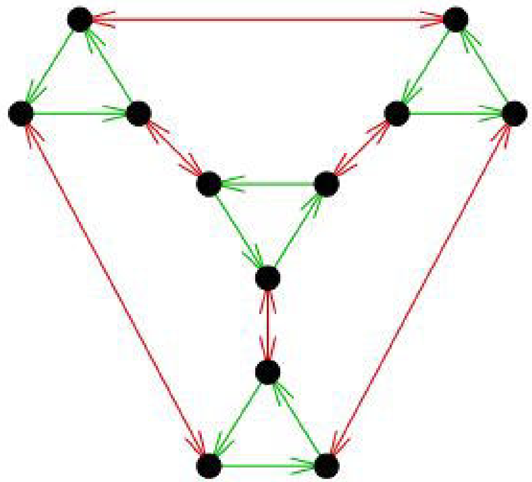
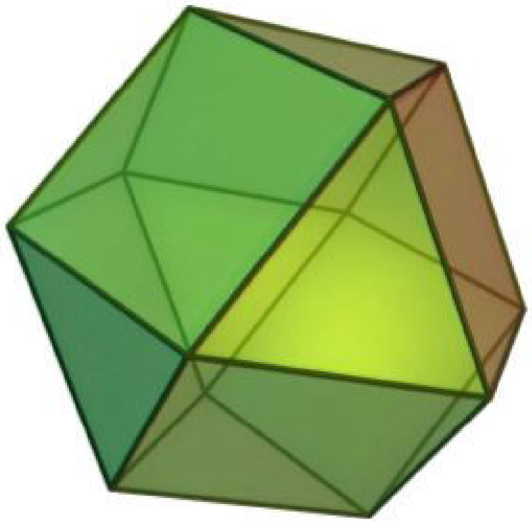
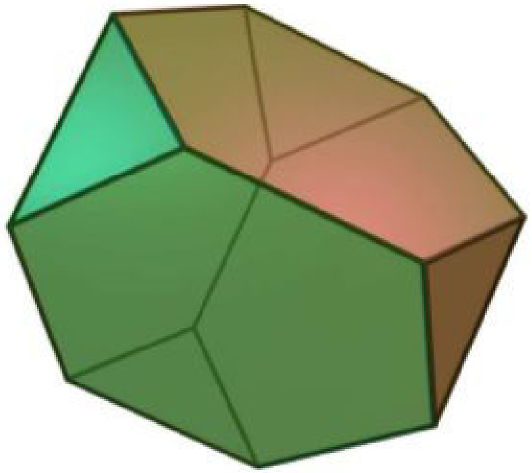


Cuboctahedron



Truncated Tetrahedron





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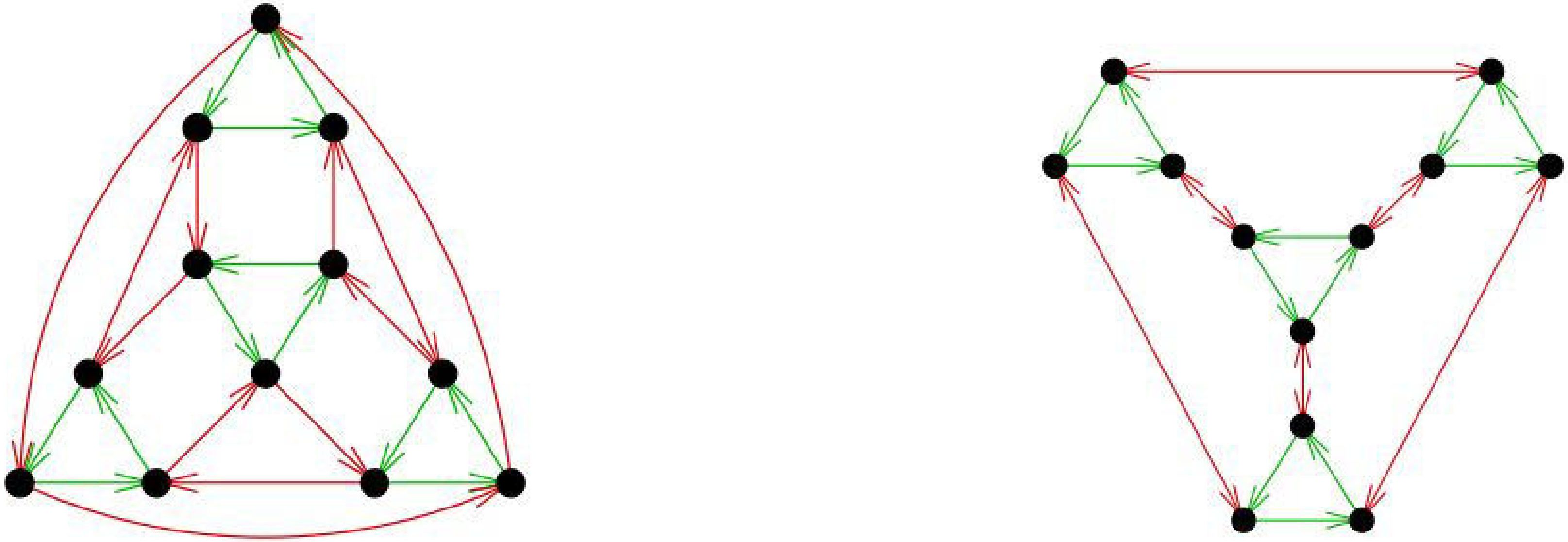


This was Cayley's original theorem (direction and color preserving graph automorphisms).
Later R. Frucht constructed (undirected and uncolored) graphs with the same automorphism group.
Again later, Z. Hedrlin, A. Pultr and finally P. Hell generalized this to monoids.

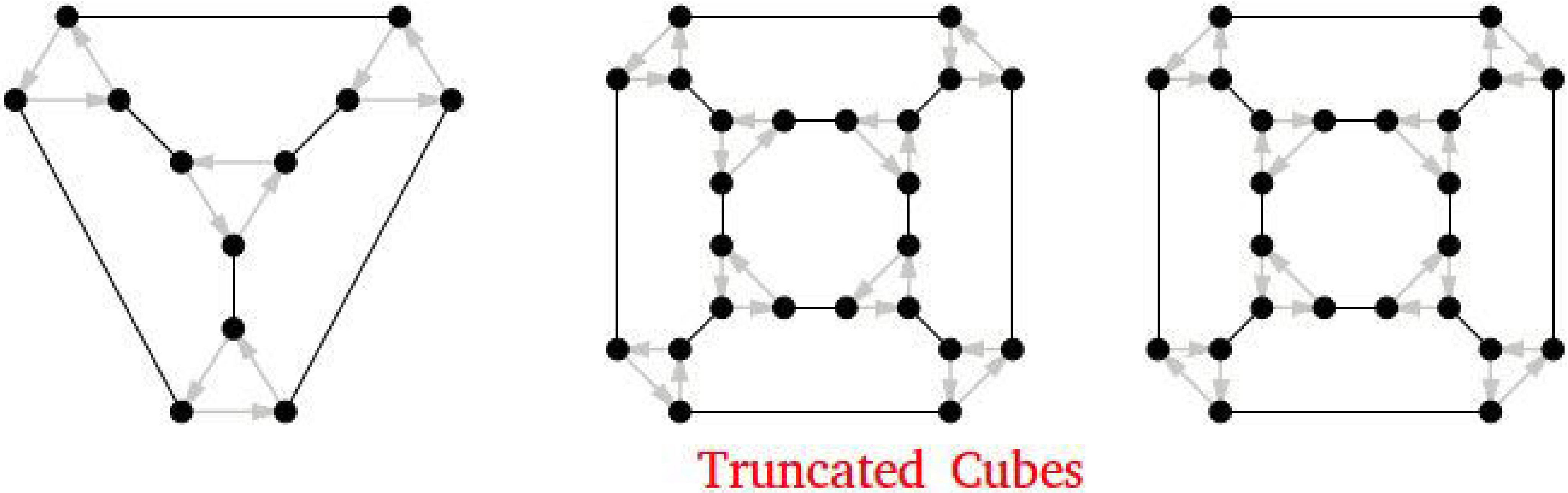
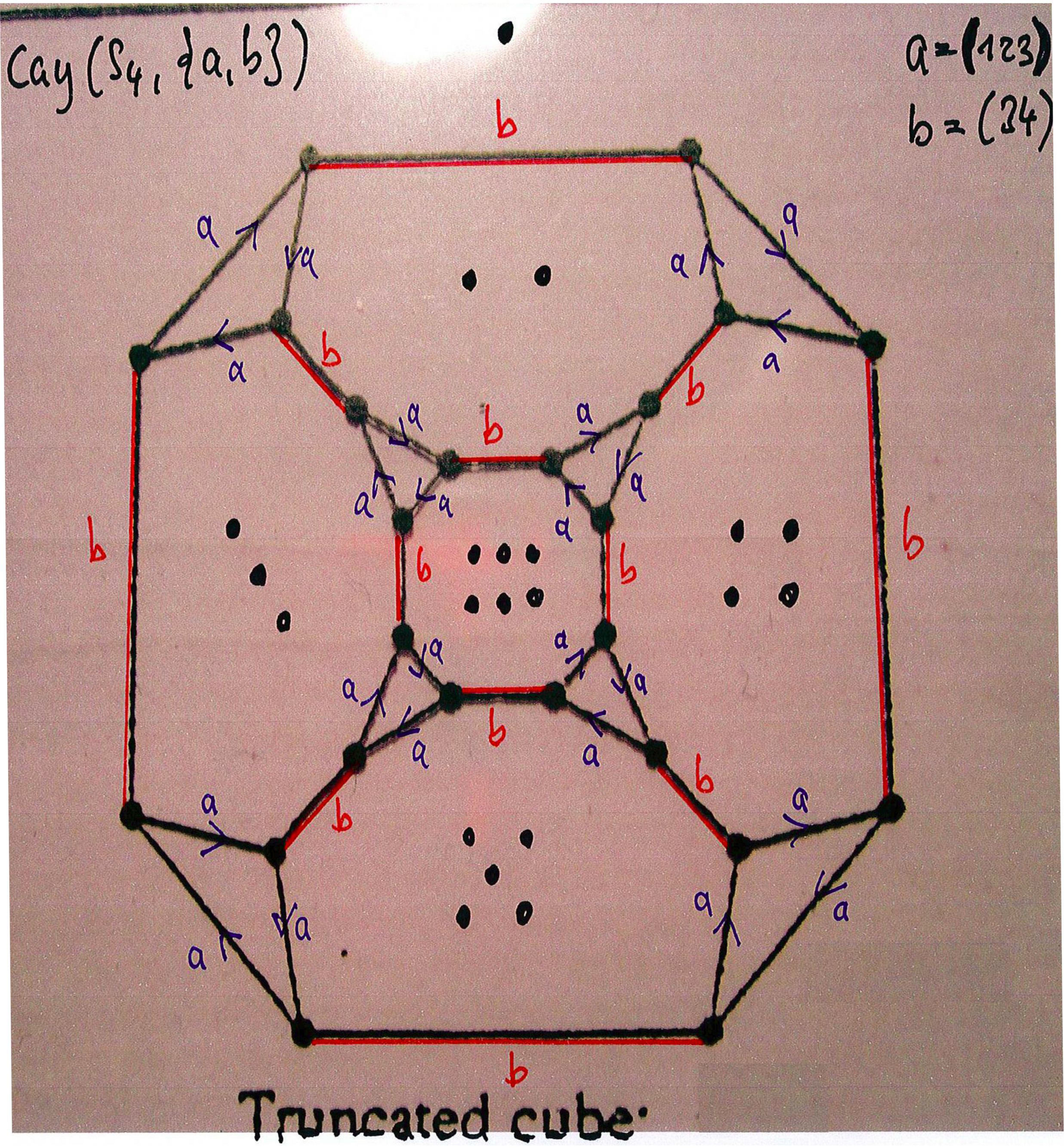
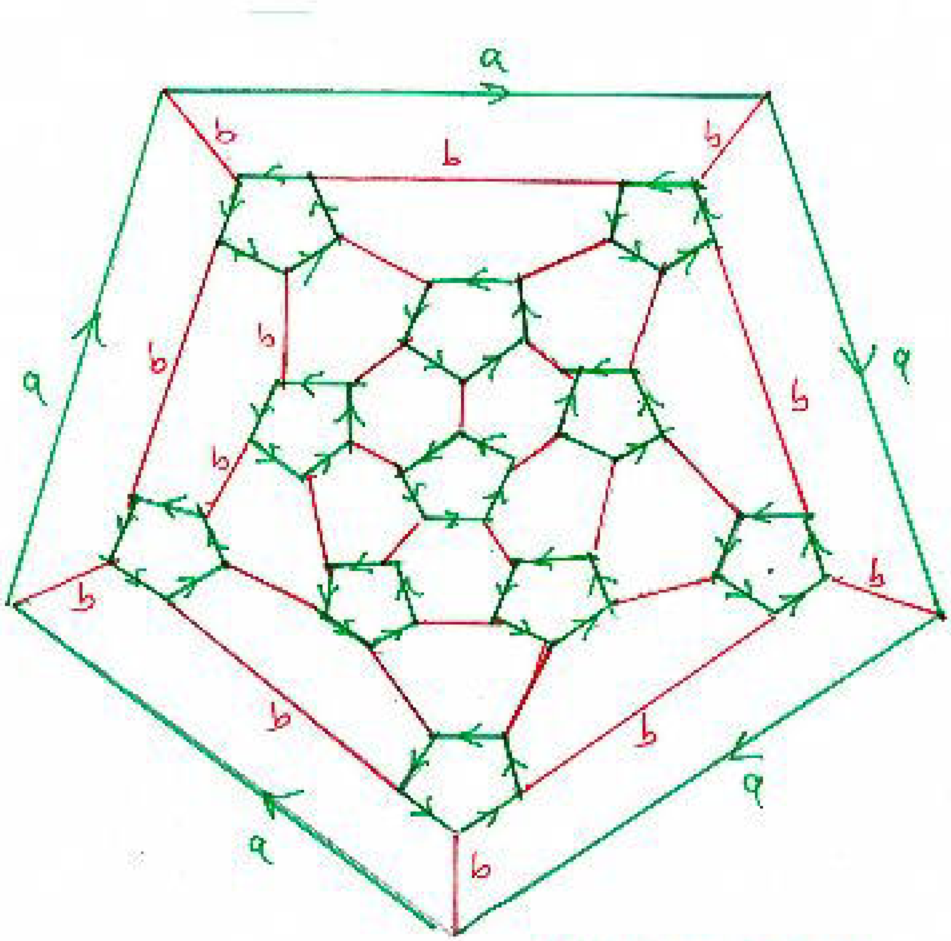


Figure 1: From left to right: the plane Cayley graphs $\mathrm{Cay}(A_4, \{(12)(34), (123)\})$, $\mathrm{Cay}(S_4, \{(123), (34)\})$, and $\mathrm{Cay}(\mathbb{Z}_2 \times A_4, \{(0, (123)), (1, (12)(34))\})$.



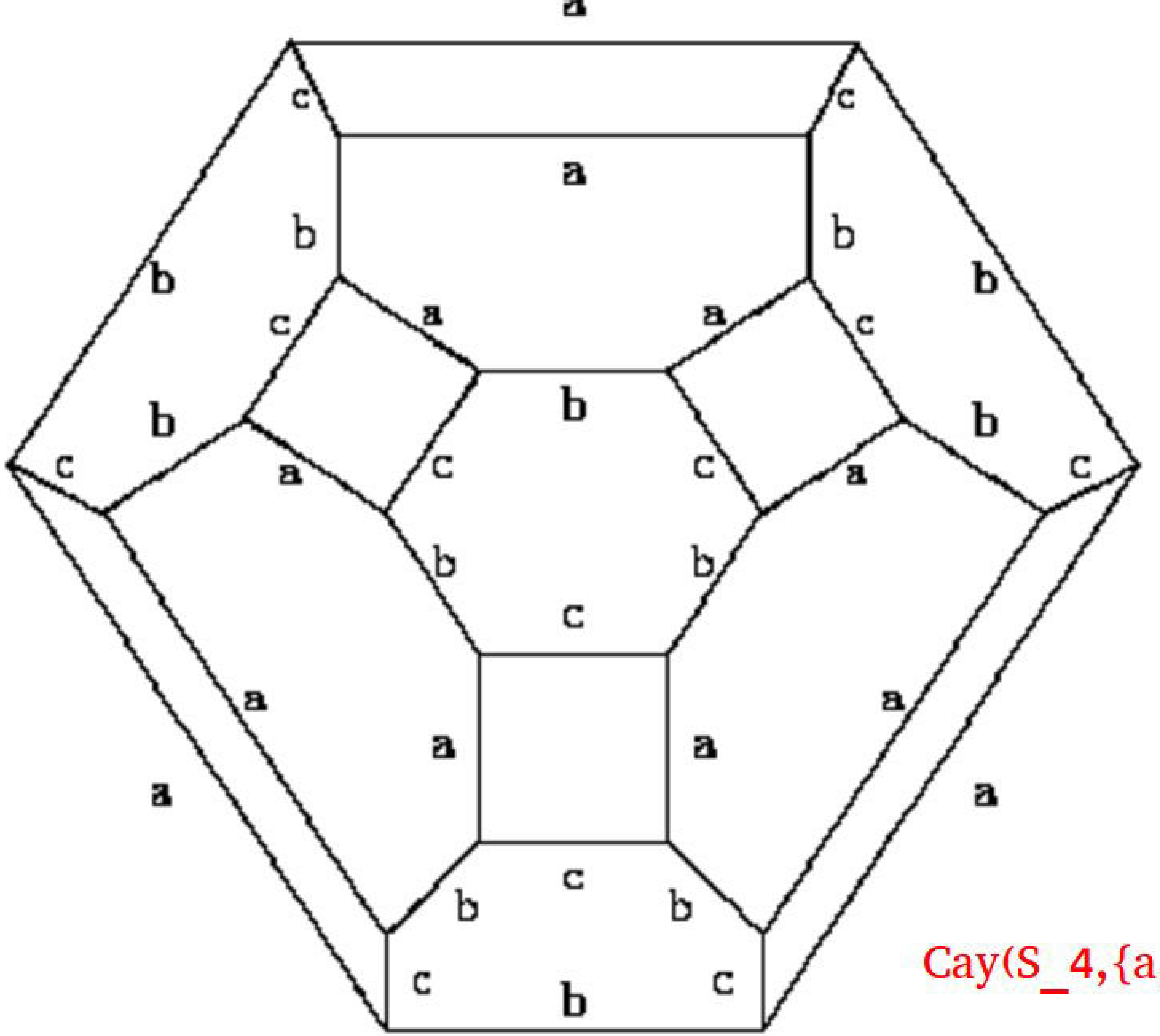
All automorphisms are color preserving



Cay(A_5,{a,b})

Truncated Icosahedron

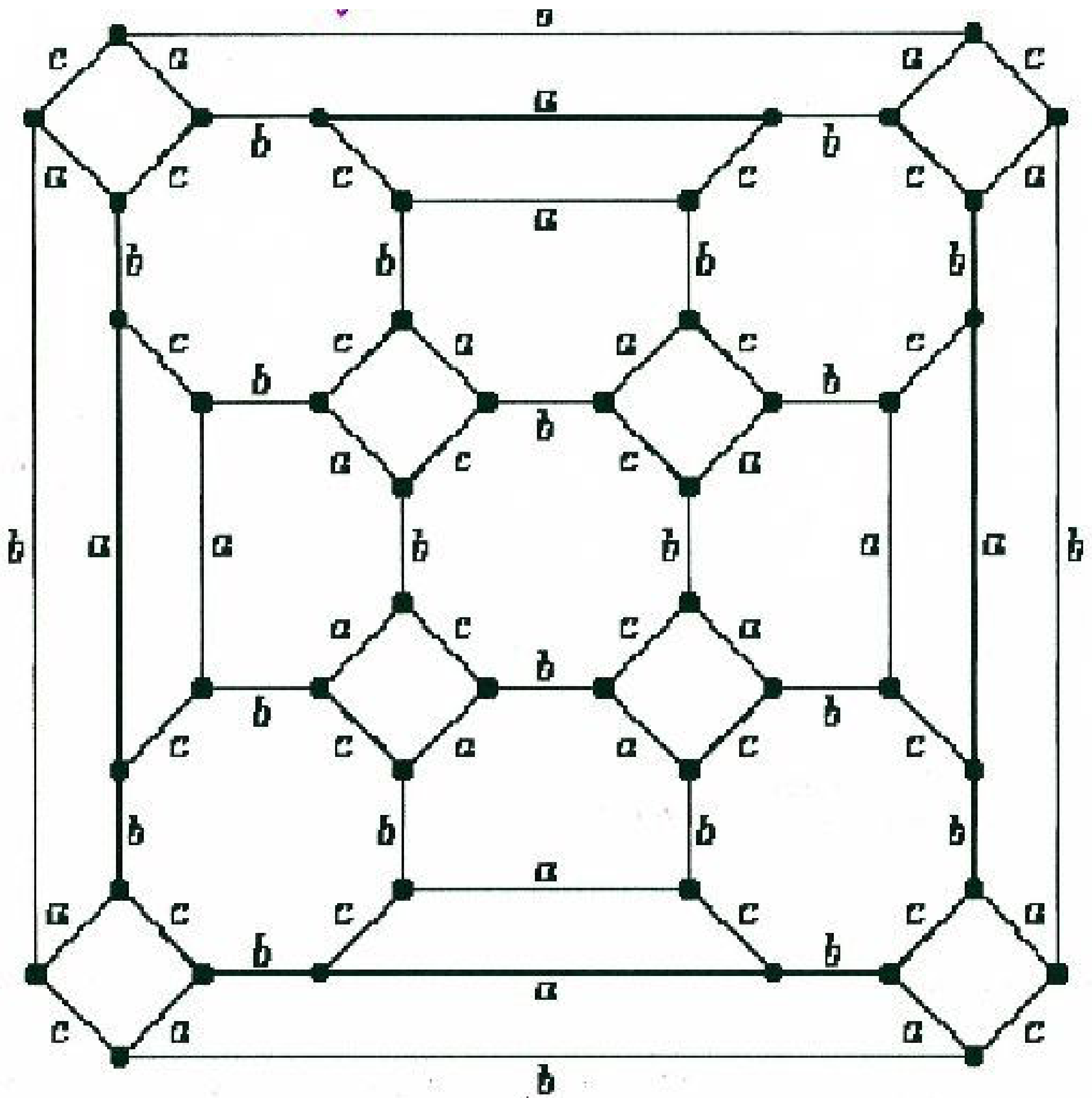
All automorphisms are color preserving



Cay(S_4,{a,b,c})

Truncated Octahedron

Figure 1: The Cayley graph of the symmetries of the tetrahedron

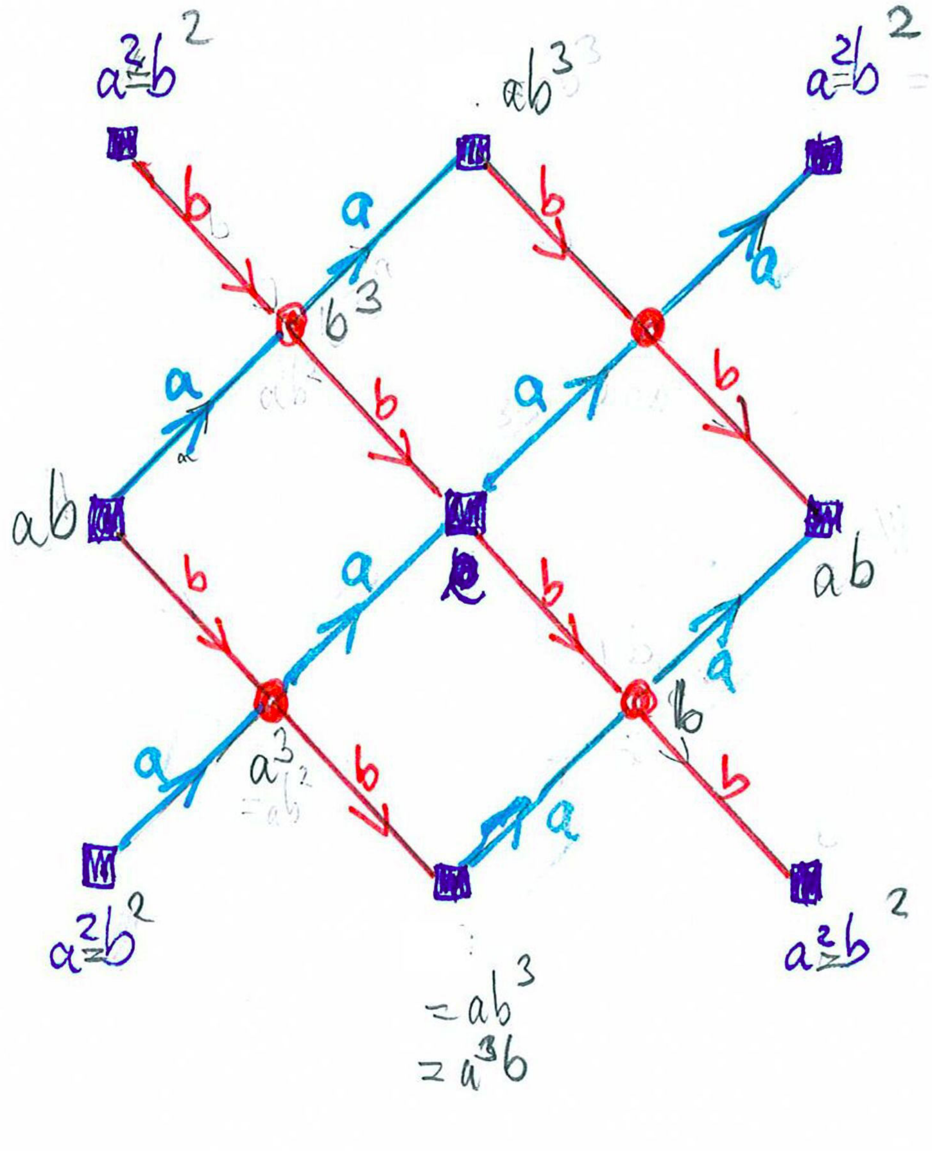


: The Cayley graph of the symmetries of the cube.

Great Rhombicuboctaheron

Cay(Z_2 x S_4,{a.b.c})

Quaternions $Q=\{a,b;a^4=b^4=(ab)^2=e\}$



K4,4

A.T. White, Graphs and groups on surfaces, Elsevier 2001:

group	genus
Finite planar abelian iff:	
$Z_n, Z_2 \times Z_{2n}, (Z_2)^3$	0
$(Z_2)^4$	1
$(Z_2)^5$	5
$(Z_2)^n$, n>1	$1+ (n-4)2^{(n-3)}$
$Z_n \times D_n$, n >1 odd	1
$(Z_3)^3$	7
S_5	4
S_n, n >167	1+n!/168
A_n, n >167	< 1+n!/336
$(Z_2)^n \times Q$	$n2^n +1$
$(Z_2)^n \times Q \times Z_m$, m odd	$mn2^n +1$

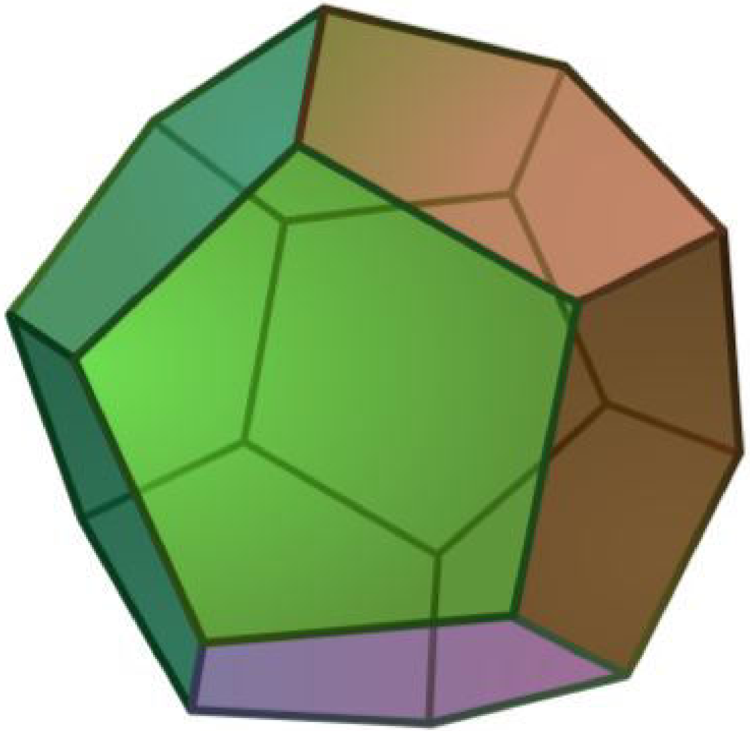
Automorphism group of the generalized Petersen graph G(8,3), which has 96 elements, has 2

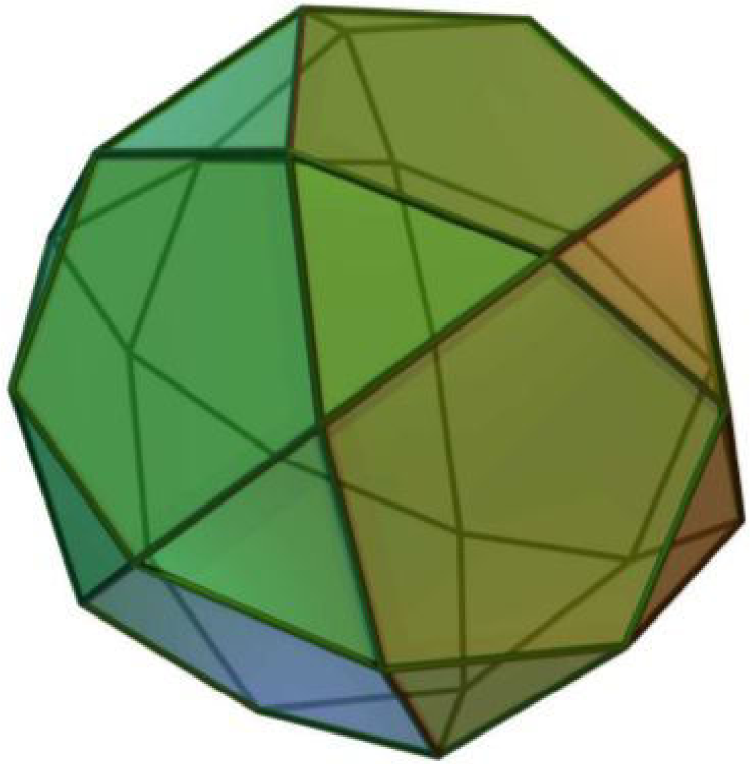
$\{a,b,c;a^2=b^2=c^2=(ab)^2=(bc)^3=(ac)^8=b(ac)^4b(ac)^4=e\}$

The smallest (by order) groups with unknown genus are non-abelian with 32 elements

. . . actually

planar Cayley graphs of these groups are exactly graphs of Platonic and Archimedian solids but the Dodecahedron and the Icosidodecahedron.





# elements	groups	#semigroups
1	1	1
2	Z_2	4
3	Z_3	18
4	Z_4, $Z_2 \times Z_2$	126
5	Z_5	1.160
6	Z_6, $D_3=S_3$	15.973
7	Z_7	836.021
8	Z_8, $(Z_2)^3$, $Z_2 \times Z_4$, D_4, Q	1.843.120.128
9	Z_9, $Z_3 \times Z_3$	? [Grillet 1996]

n	2	3	4	5	6	7	8
All	4	18	126	1 160	15 973	836 021	1 843 120 128
Commutative	3	12	58	325	2 143	17 291	221 805

n	9	10
Commutative	11 545 843	3 518 930 337
Commutative Clifford	25 284	161 698

There are so many Semigroups!

right groups

right zero band

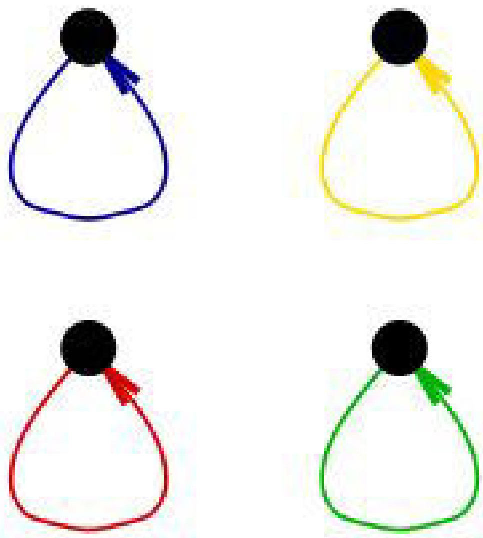
$R_n := < r_1, \ldots, r_n \mid xy = y >$

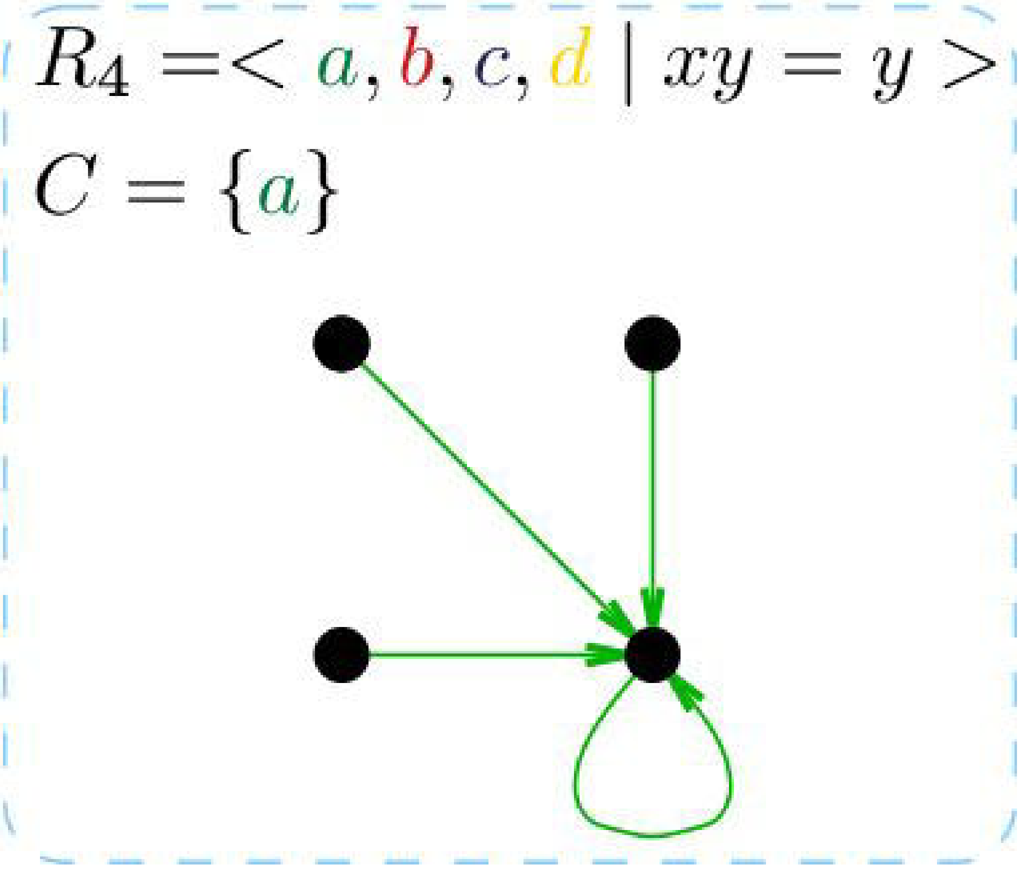
right group

$G \times R_n$ for some group G

$$L_4 = < a, b, c, d \mid xy = x >$$

$$C = \{a, b, c, d\}$$



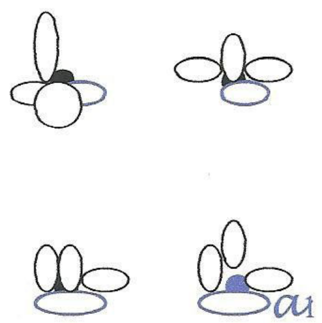

$R_4 = < a, b, c, d \mid xy = y >$
$C = \{a\}$

Planar
Right Cayley Graphs
of
Left zero Right zero
Semigroups

$L_4 =$

$< a_1, a_2, a_3, a_4 \mid a_i a_j = a_i >$

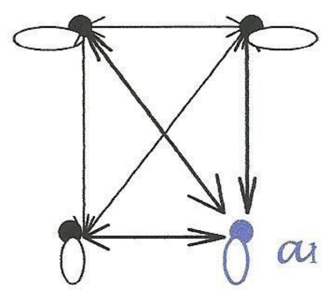
$C = \{a_1, a_2, a_3, a_4\}$



$R_4 =$

$< a_1, a_2, a_3, a_4 \mid a_i a_j = a_j >$

$C = \{a_1, a_2, a_3, a_4\}$

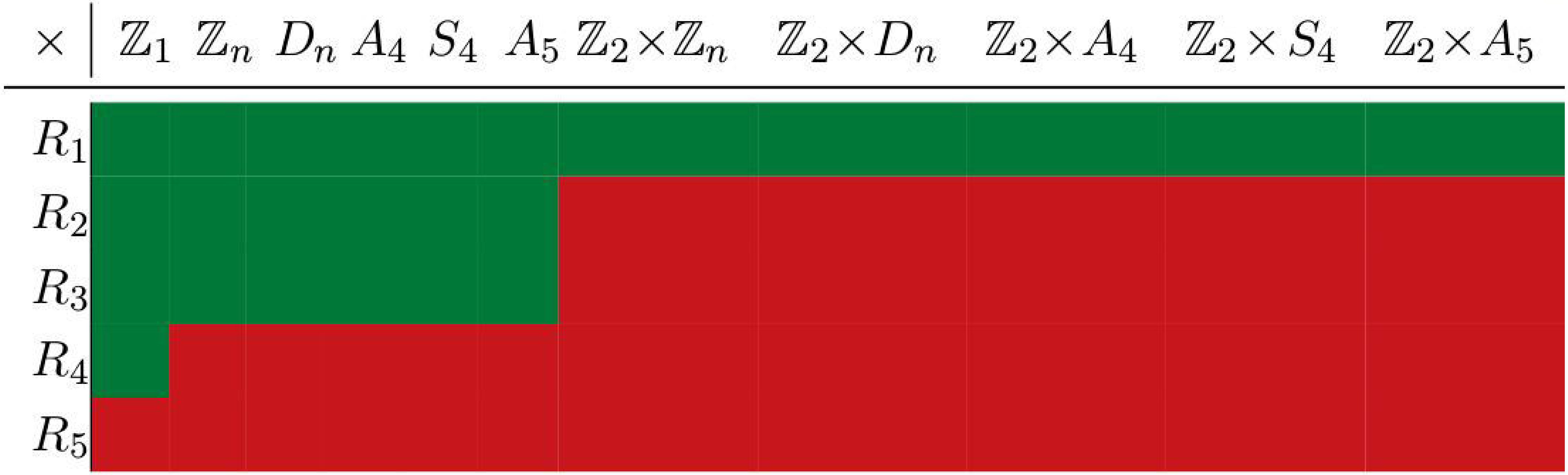


So all left zero semigroups are planar,
all right zero semigroups starting from R_5 are not planar

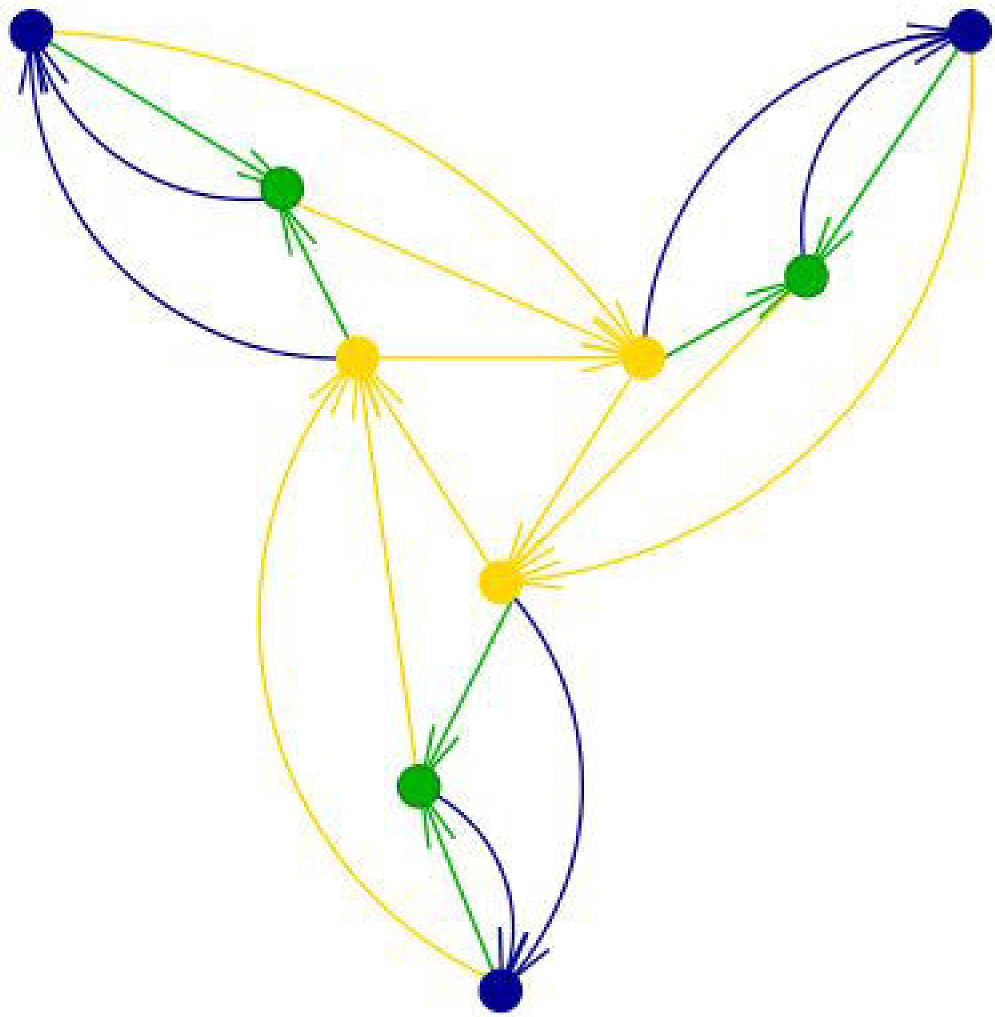
Cay(R_5,{a_1,a_2.a_3,a_4,a_5}) is K_5 with loops

our result:

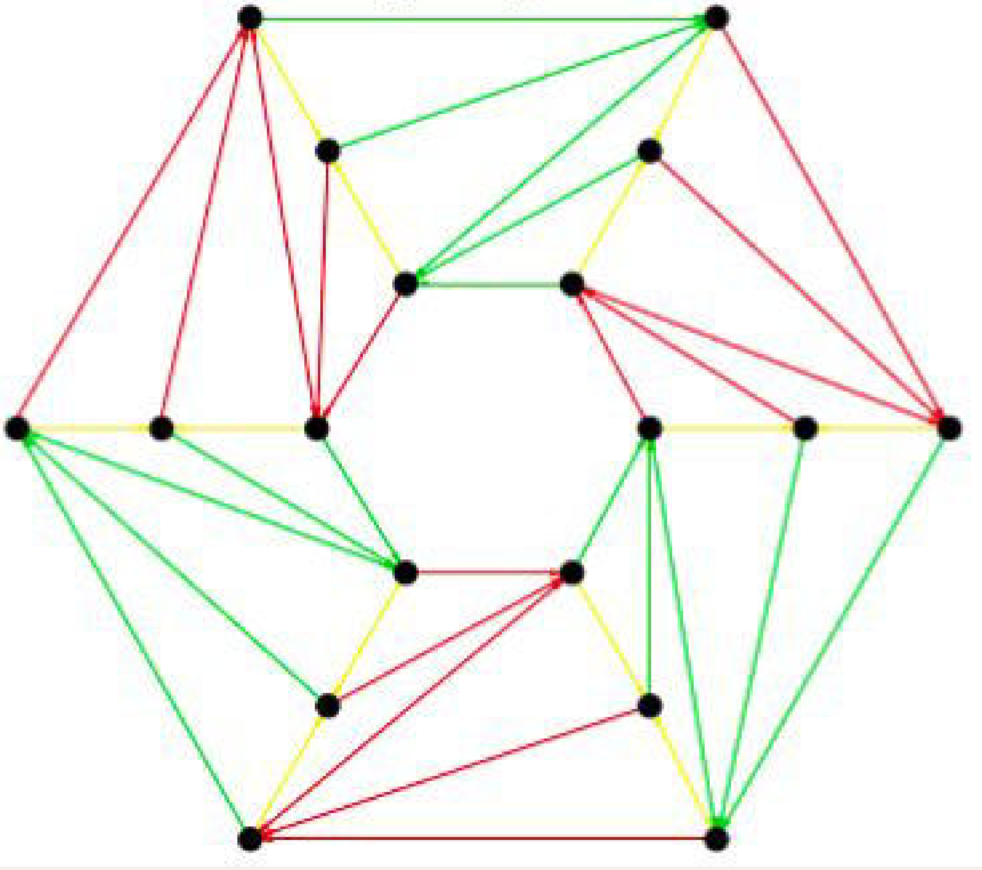
the planar right groups are exactly the green ones

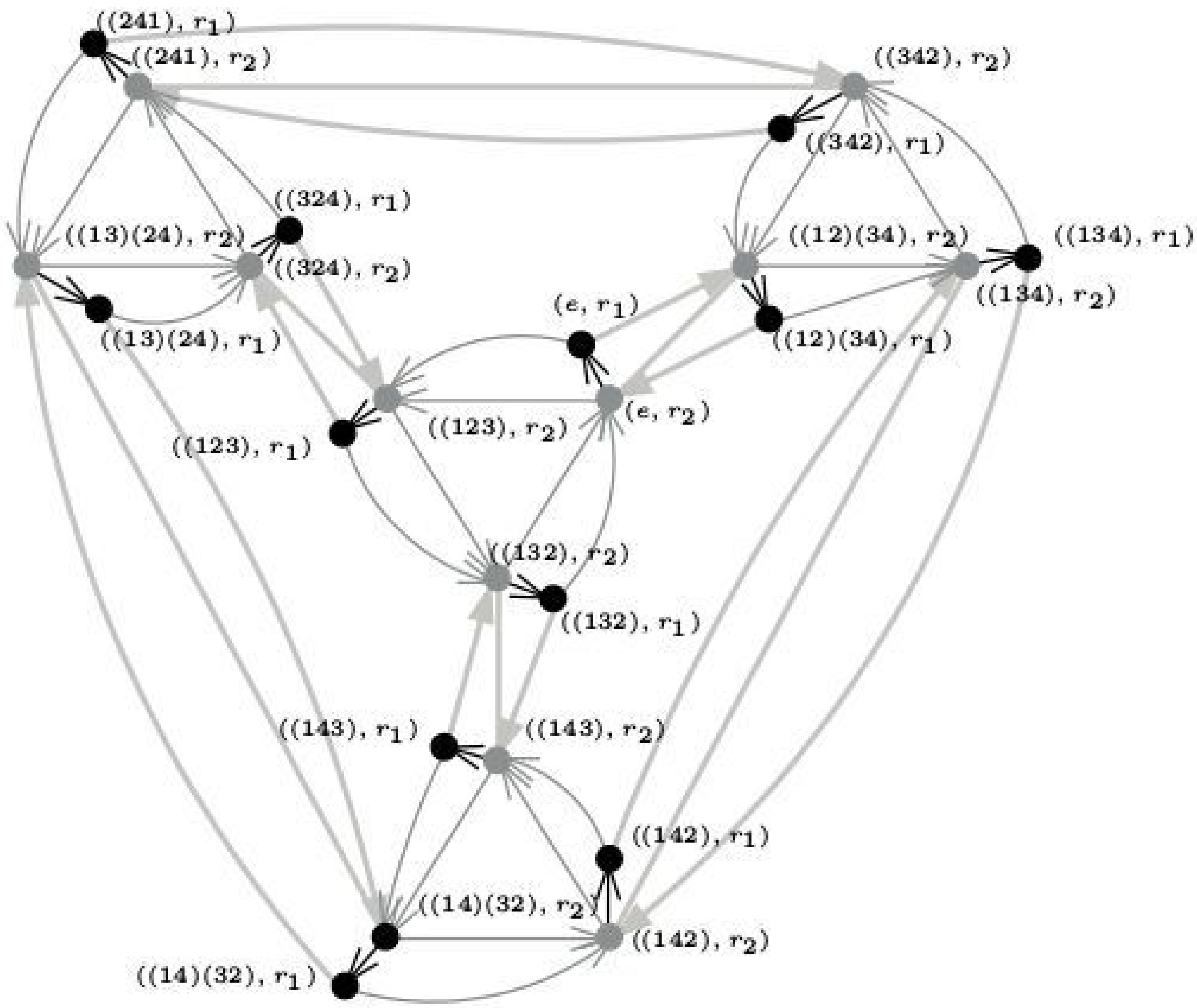


Cay(Z_3 x R_3,{(1,r_1),(0,r_2),(0,r_3)}

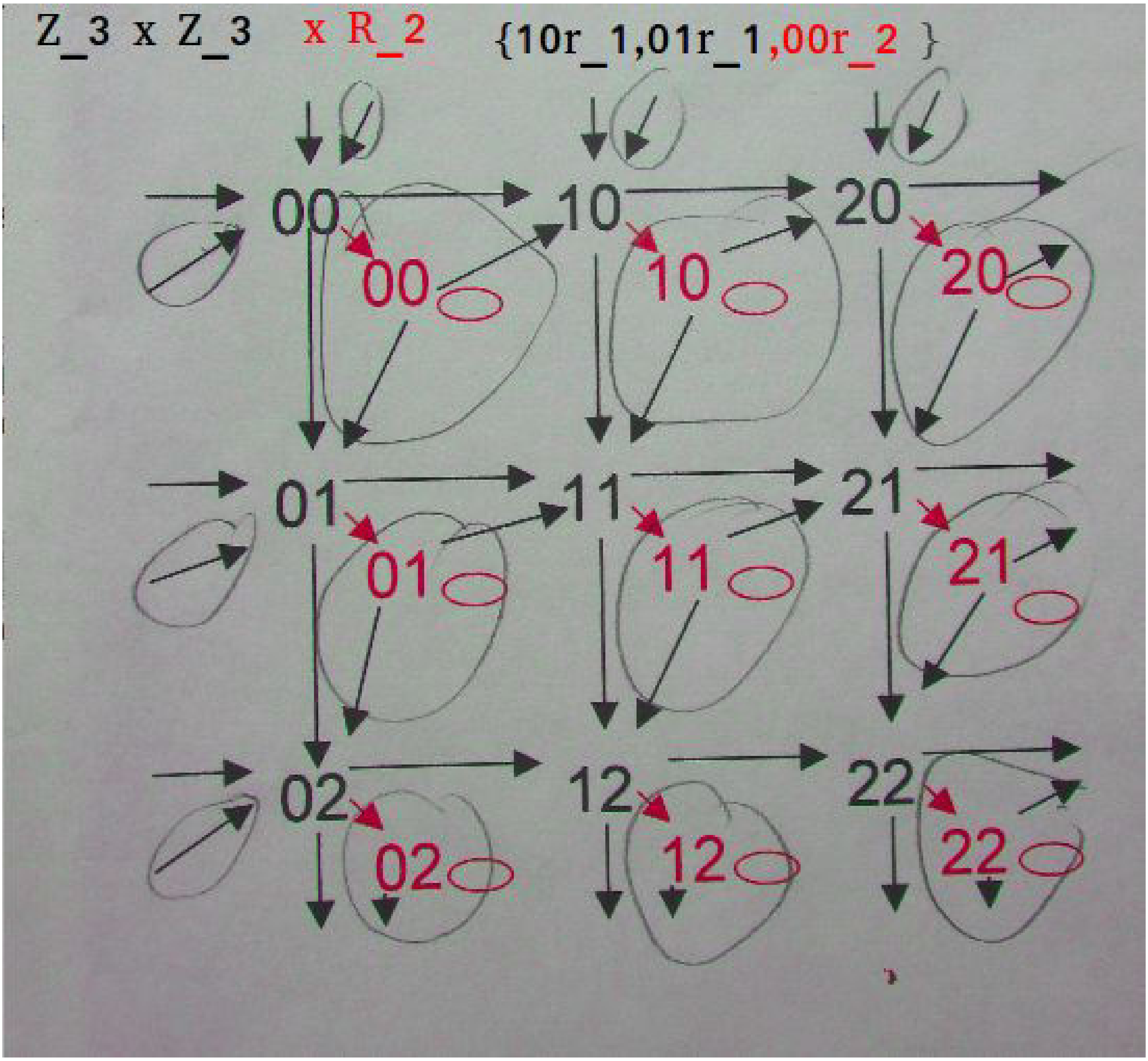


D_3 x R_3





igure 6: The graph Cay$(A_4 \times R_2, \{(e, r_1), ((12)(34), r_2), ((123), r_2)\})$.


Z_3 x Z_3 x R_2 {10r_1,01r_1,00r_2 }
00
10
20
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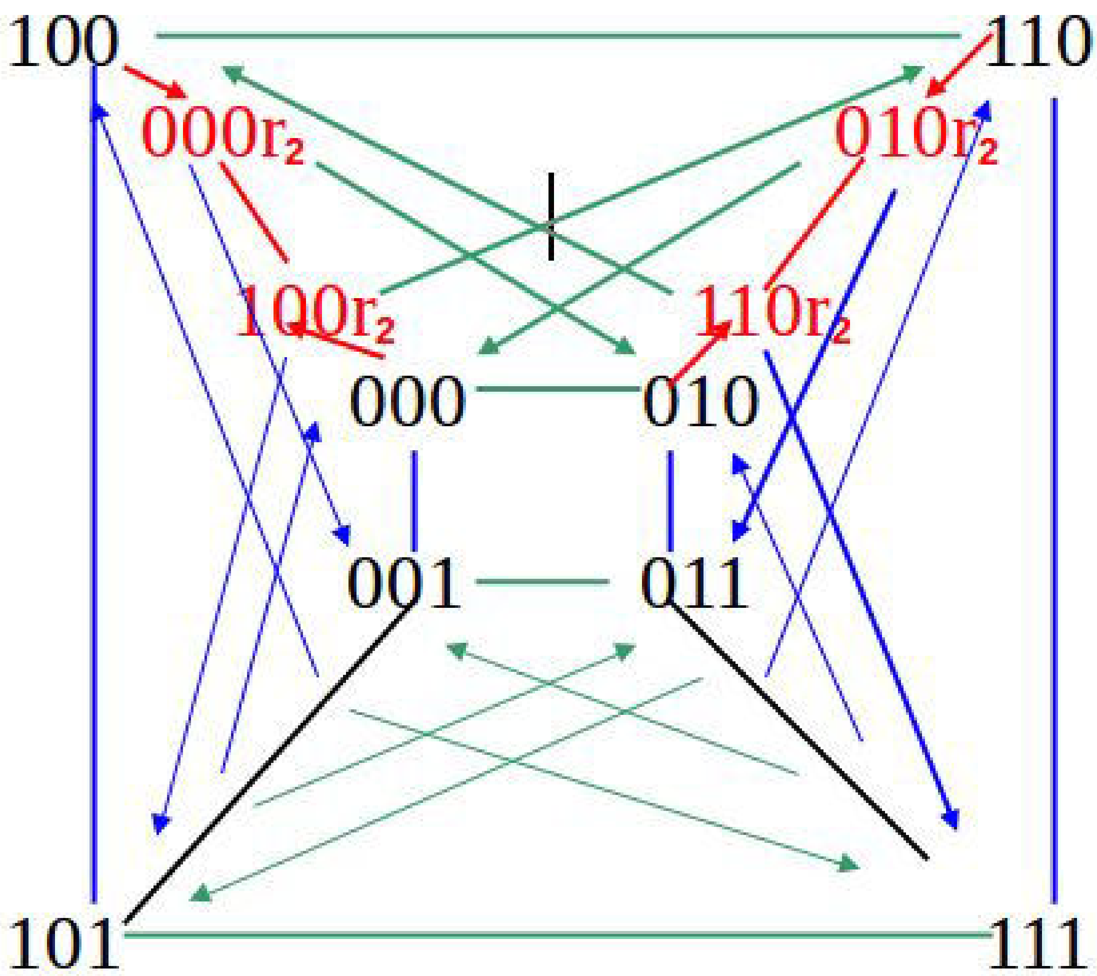

$$Z_2 \times Z_2 \times Z_2 \times R_2 = Z_2 \times D_2 \times R_2$$

Schwarz wechselt vorn, grün wechselt in der Mitte, blau wechselt hinten
Wenn man das für jede Diagonale so ersetzt, reicht in jedem Trapez eine Brücke, also Genus wird höchstens 2n bei D_{2n}

further remarks

characterization of finite planar semigroups?

- are dodecahedron and icosidodecahedron cayley graphs of semigroups?

- is there anything in the flavour of discrete isometry *semigroups* of the sphere?

characterization of cayleygraphs of semigroups?

- in groups: Γ is cayley graph iff there is $G < \mathrm{Aut}(\Gamma)$ acting fixpoint-free and for all $v, w \in V$ there is $g \in G$ with $g(v) = w$.

- what are necessary and sufficient conditions for semigroups? work in progress..

- given $S < T$ is some cayley graph of S a minor of $\mathrm{Cay}(T, C)$?

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