Construction of a complement to a quasiorder

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5.2. – 7.2.2016

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- **the lattice** $\mathrm{Quord}(A, \subseteq)$ **of all quasiorders of an algebra** $\mathcal{A}$

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- M. Erné and J. Reinhold (1995): lattices of all quasiorders on a **set**
  - atomistic
  - dually atomistic
  - complemented
- I. Chajda and G. Czédli (1996), A. G. Pinus (1995):
  - every algebraic lattice is isomorphic to the quasiorder lattice of a suitable algebra
- G. Czédli and A. Lenkehegyi (1983), A. G. Pinus and I. Chajda (1993):
  - quasiorder lattice of a majority algebra is always distributive
- R. Pöschel and S. Radeleczki:
  - how endomorphisms of quasiorders behave
  - when $\mathrm{End}\ q \subseteq \mathrm{End}\ q^{'}$ for quasiorders $q, q^{'}$ on a set A ($\mathrm{End}\ q$ is the set of all mappings preserving q)
  - description of the quasiorder lattice of the algebra $(A, \mathrm{End}\ q)$

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- **an element** $x \in A$ is referred to as **cyclic** if there exists a positive integer n such that $f^n(x) = x$

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- Find necessary and sufficient conditions for a monounary algebra (A, f), under which the lattice $\mathrm{Quord}\,(A, f)$ is complemented.
- Construct a complementary quasiorder to a given quasiorder, if the lattice $\mathrm{Quord}(A, f)$ is complemented.

Result

Theorem

Let (A, f) be a monounary algebra. The lattice $\mathrm{Quord}\,(A, f)$ is complemented if and only if

- *each connected component of (A, f) contains a cycle,*
- *there is $n \in N$ such that each cycle of (A, f) has n elements,*
- *n is square-free,*
- *for each $a \in A$, the element $f(a)$ is cyclic.*

Sufficiency of the condition was proved by means of transfinite induction. We will describe a **construction of a complement** to a given quasiorder of (A, f) satisfying this condition.

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  - for each $a \in A$, the element $f(a)$ is cyclic.
- Let $\alpha \in \mathrm{Quord}\,(A, f)$.
- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.

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- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.

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- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \dots, u_k = b$ elements of A' such that $(\forall i \in \{0, \dots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha})$.

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  1. $(\forall x \in D \setminus P(D))(\exists y \in P(D))((x, y) \in \alpha, (y, x) \in \alpha)$;
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Let $x, y \in A$. We put $(x, y) \in \beta$ if either $x = y$ or (x, y) fulfills one of the steps of the construction (K).

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.
- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.
- *Step (c).* Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.
- *Step (d1).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

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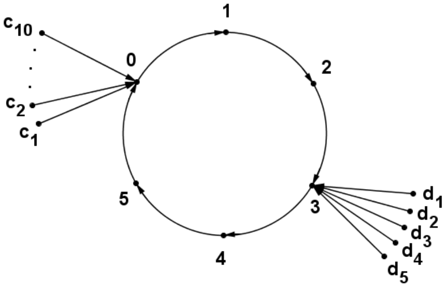
- *Step (d'1).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k.$
- *Step (d2).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.
- *Step (d'2).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(x, p(D)) \in \alpha$.
- *Step (e).* Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

Construction (K) - example

Let (A, f) be a given algebra:



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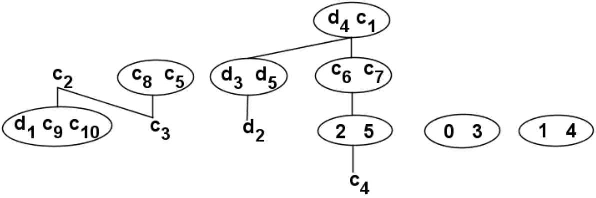

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$c_2, c_3, c_5, c_8, c_9, c_{10}$	d_2, d_3, d_5	d_1

- A'/ρ :

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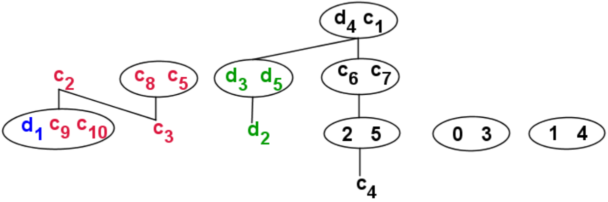

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Let:

- $P(D_1) = \{c_2, c_3, c_5, c_9\}$ and $p(D_1) = c_2$
- $P(D_2) = \{d_2, d_3\}$ and $p(D_2) = d_2$
- $P(D_3) = \{d_1\}$ and $p(D_3) = d_1$

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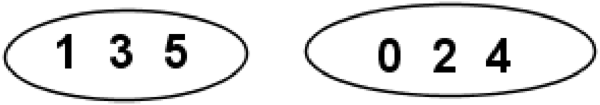

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Step (b). Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

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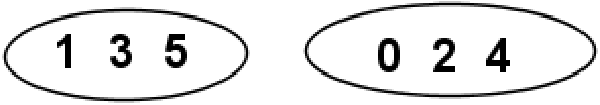

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Step (c). Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

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- We distinguish three cases:
  1. $x, y \in P(D_1) = \{c_2, c_3, c_5, c_9\}$, then $(x, y) \in \beta$ if and only if $(x, y) \in \{(c_2, c_3), (c_2, c_9), (c_5, c_3)\}$,

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  3. $x, y \in P(D_3) = \{d_1\}$, then $(x, y) \in \beta$.

Construction (K) - example

Step (c). Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

- We distinguish three cases:
  1. $x, y \in P(D_1) = \{c_2, c_3, c_5, c_9\}$, then $(x, y) \in \beta$ if and only if $(x, y) \in \{(c_2, c_3), (c_2, c_9), (c_5, c_3)\}$,
  2. $x, y \in P(D_2) = \{d_2, d_3\}$, then $(x, y) \in \beta$ if and only if $(x, y) = (d_3, d_2)$,
  3. $x, y \in P(D_3) = \{d_1\}$, then $(x, y) \in \beta$.

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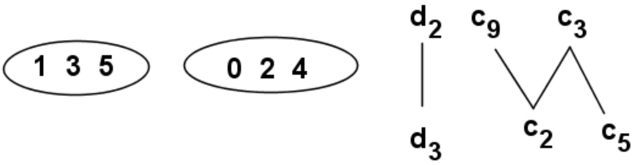

Step (d1). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if
$(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k.$

Construction (K) - example

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- $x \in \{0, 1, 2, 3, 4, 5\}, y \notin A'$ i.e. $y \in \{c_1, c_4, c_6, c_7, d_4\}$.

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Construction (K) - example

Step (d1).

- It follows that $(x, y) \in \beta$ if and only if $(x, y) \in \{(1, c_4), (3, c_4), (5, c_4)\}$.

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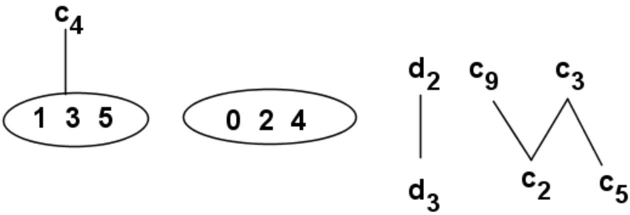

Construction (K) - example

Step (d'1). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if
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- $x \notin A'$ i.e. $x \in \{c_1, c_4, c_6, c_7, d_4\}$, $y \in \{0, 1, 2, 3, 4, 5\}$.

Construction (K) - example

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- $\alpha \restriction C = \theta_3, e = 2.$
- We set $(x, y) \in \beta$ if and only if $(f^6(y), y) \in \alpha$, $(y, f^6(y)) \notin \alpha$, $x = f^k(y), e/k$ i.e. $k \equiv 0 \pmod 2$.

Construction (K) - example

Step (d'1).

- It follows that $(x, y) \in \beta$ if and only if either
  $x \in \{c_1, c_6, c_7\} \wedge y \in \{1, 2, 3\}$, or
  $(x, y) \in \{(d_4, 0), (d_4, 2), (d_4, 4)\}$.

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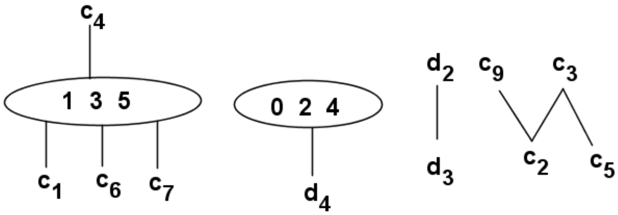

Construction (K) - example

Step (d2). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.

Construction (K) - example

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- $x \in \{0, 1, 2, 3, 4, 5\}, y \in A'$ i.e. $y \in \{c_2, c_3, c_5, c_8, c_9, c_{10}, d_1, d_2, d_3, d_5\}$.

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Construction (K) - example

Step (d2).

- It follows that $(x, y) \in \beta$ if and only if either $x \in \{1, 3, 5\}$ and $y \in \{c_2, c_3, c_9\}$, or $x \in \{0, 2, 4\}$ and $y \in \{d_1, d_2\}$.

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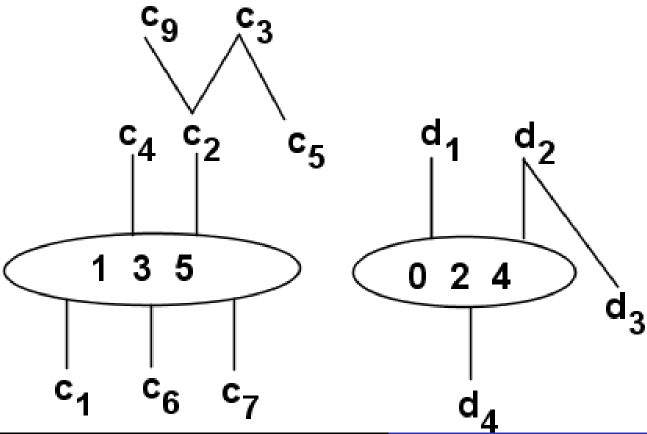

Step (d'2). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(x, p(D)) \in \alpha$.

- $x \in A'$ e.i. $x \in \{c_2, c_3, c_5, c_8, c_9, c_{10}, d_1, d_2, d_3, d_5\}$, $y \in \{0, 1, 2, 3, 4, 5\}$.

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- We set $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), k \equiv 0$ (mod 2) and $(p(D), x) \in \alpha$.

Construction (K) - example

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- It follows that $(x, y) \in \beta$ if and only if either $x = c_2$ and $y \in \{1, 3, 5\}$ or $x \in \{d_1, d_2, d_3\}$ and $y \in \{0, 2, 4\}$.

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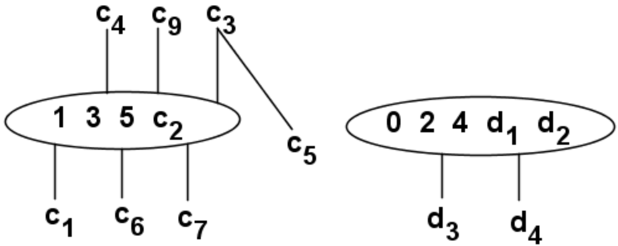

Construction (K) - example

Step (e). Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

- In this example, remaining cases are:

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- Then $(x, y) \in \beta$ if and only if $(x, f^6(x)) \in \beta$, $(f^6(x), f^6(y)) \in \beta$, $(f^6(y), y) \in \beta$.

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1. If $x \in P(D_i), y \in P(D_j)$, $i \neq j$, $i, j \in \{1, 2, 3\}$, then $(x, y) \in \beta$ if and only if $x \in \{d_1, d_2\}$ and $y \in \{d_1, d_2\}$.

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2. If x, y are noncyclic elements such that $x, y \notin P(D)$ for any $D \in A'/\rho$, then $(x, y) \in \beta$ if and only if either $x \in \{c_1, c_6, c_7, c_2\}$ and $y = c_4$, or $x \in \{c_1, c_6, c_7\}$ and $y = c_2$.

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Step (e). $(x, y) \in \beta$ if and only if $(x, f^6(x)) \in \beta$, $(f^6(x), f^6(y)) \in \beta$, $(f^6(y), y) \in \beta$. It follows that

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3. If $x \in P(D)$ for some $D \in A'/\rho$ and y is noncyclic element such $y \notin P(D)$ for any $D \in A'/\rho$, then $(x, y) \in \beta$ if and only if $x = c_2, y = c_4$.

Construction (K) - example

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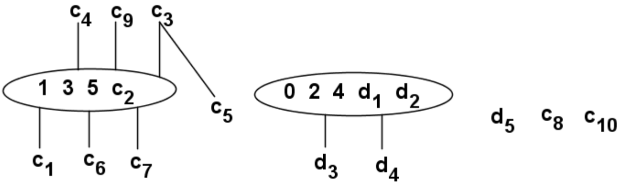
3. If $x \in P(D)$ for some $D \in A'/\rho$ and y is noncyclic element such $y \notin P(D)$ for any $D \in A'/\rho$, then $(x,y) \in \beta$ if and only if $x = c_2, y = c_4$.
4. If x is noncyclic element such $x \notin P(D)$ for any $D \in A'/\rho$ and $y \in P(D)$ for some $D \in A'/\rho$, then $(x,y) \in \beta$ if and only if either $x \in \{c_1, c_6, c_7\}$ and $y \in \{c_2, c_3, c_9\}$, or $x = d_4$ and $y \in \{d_1, d_2\}$.

We constructed a complementary quasiorder β to the quasiorder α.



Complementarity - main result

Hypothesis

Let (A, f) *be a monounary algebra whose lattice* $\mathrm{Quord}\,(A, f)$ *is complemented. Let* $\alpha \in \mathrm{Quord}\,(A, f)$.
A relation β *on* A *is a complement in* $\mathrm{Quord}\,(A, f)$ *to* α *if and only if* β *is constructed by the construction (K).*

Thank you for your attention.