Sharp and principal elements in effect algebras

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Definition

An *effect algebra* is defined to be an algebraic system $(E, 0, 1, \oplus)$ consisting of a set E, two special elements $0, 1 \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on E that satisfies the following conditions for all $p, q, r \in E$:

1. [Commutative Law] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

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1. [Commutative Law] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
2. [Associative Law] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

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3. [Orthosupplementation Law] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

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3. [Orthosupplementation Law] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = 1$.
4. [Zero-unit Law] If $1 \oplus p$ is defined, then $p = 0$.

basic properties

If $p, q \in E$, we say that p and q are orthogonal and write $p \perp q$ iff $p \oplus q$ is defined in E. If $p, q \in E$ and $p \oplus q = 1$, we call q the *orthosupplement* of p and write $p' = q$.

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It is shown in [5] that the relation $\leq$ defined for $p, q \in E$ by $p \leq q$ iff $\exists r \in E$ with $p \oplus r = q$ is a partial order on E and $0 \leq p \leq 1$ holds for all $p \in E$. It is also shown that the mapping $p \mapsto p'$ is an order-reversing involution and that $q \perp p$ iff $q \leq p'$. Furtheremore, E satisfies the following *cancellation law*: If $p \oplus q \leq r \oplus q$, then $p \leq r$.

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An element $a \in E$ is *sharp* if $a \wedge a' = 0$.

We denote the set of sharp elements of E by S_E.

An element $a \in E$ is said to be *principal* iff for all $p, q \in E$, $p \perp q$ and $p, q \leq a \Rightarrow p \oplus q \leq a$. We denote the set of principal elements of E by P_E.

Technical tools

In order to characterize principal elements we need the following known Theorem:

Theorem (1, Theorem 3.5 in 8)

If $p, q \in E$, $p \perp q$, *and* $p \vee q$ *exists in* E, *then* $p \wedge q$ *exists in* E, $p \wedge q \leq (p \vee q)' \leq (p \wedge q)'$ *and* $p \oplus q = (p \wedge q) \oplus (p \vee q)$.

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Lemma (2)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. If* $x \in P_E$, $t \in E$ *and* $t \leq x$ *then there exists* $t \vee x'$ *in* E *and*

$$t \vee x' = t \oplus x'.$$

Proof

Suppose that $x \in P_E$. Let $t \in E$ and $t \leq x$ hence $t \perp x'$. We show that $t \oplus x'$ is the join of t and x'.
Obviously $t \leq t \oplus x'$ and $x' \leq t \oplus x'$. Suppose that $u \in E$, $t \leq u$ and $x' \leq u$ then

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$$u' \leq x \quad t \leq x. \tag{2}$$

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Now (1) and (2) implies $t \oplus u' \leq x$ since $x \in P_E$. Hence $x' \perp (t \oplus u')$ and by associativity $x' \perp t$ and $(x' \oplus t) \perp u'$ thus $t \oplus x' \leq u$ so $t \oplus x'$ is the smallest upper bound of the set $\{t, x'\}$ and therefore $t \oplus x' = t \vee x'$. $\square$

de Morgan's law

It turns out that under some conditions every effect algebra satisfies the de Morgan's law.

Lemma (3)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. If* $x, y \in E$ *and there exists* $x \vee y$ *in* E *then there exists* $x' \wedge y'$ *in* E *and*

$$x' \wedge y' = (x \vee y)'$$

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Lemma (4)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. If* $x, y \in E$ *and there exists* $x \wedge y$ *in* E *then there exists* $x' \vee y'$ *in* E *and*

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Principal elements

Theorem (5)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. Then*

$$P_E = \{x \in E\colon x \in S_E \text{ and } \forall_{t \in E}\, t \leq x \Rightarrow t \vee x' \text{ exists in } E\}$$

Proof

Suppose that $x \in P_E$ then $x \in S_E$ (see Lemma 3.3 in [8]). Let $t \in E$ and $t \leq x$. Then there exists $t \vee x'$ in E by Lemma (2).
Suppose that $x \in S_E$ and

$$\forall_{t \in E} t \leq x \Rightarrow t \vee x' \text{ exists in } E. \tag{4}$$

We show that $x \in P_E$.

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We show that $x \in P_E$.
If $u, s \in E$, $u \leq x$, $s \leq x$ and $u \perp s$ then

$$u \wedge x' = 0 \tag{5}$$

because: if $y \leq x'$ and $y \leq u \leq x$ then $y = 0$ since $x \wedge x' = 0$.
Moreover $u \leq x$ so $u \vee x'$ exists by (4). By Theorem (1).

$$u \oplus x' = (u \wedge x') \oplus (u \vee x') \overset{(5)}{=} u \vee x' \tag{6}$$

Proof-cont.

By Lemma (3) we have

$$u' \wedge x = (u \vee x')'. \tag{7}$$

Moreover $s \leq u'$ (since $u \perp s$) and $s \leq x$ so $s \leq u' \wedge x$. Hence by (6) and (7) we have

$$s \leq u' \wedge x = (u \vee x')' = (u \oplus x')'$$

so $s \perp (u \oplus x')$ and by associativity $s \oplus u \perp x'$ hence $s \oplus u \leq x$ and $x \in P_E$. $\square$

Main Theorem

In the following theorem we prove that in every effect algebra E sharp and principal elements coincide if and only if there exists in E join of every two orthogonal elements such that one of them is sharp.

Theorem (6)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. Then* $S_E = P_E$ *if and only if*

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Effect algebras closed under $\oplus$

Lemma (7)

Let $(E, 0, 1, \oplus)$ be an effect algebra. If P_E is closed under $\oplus$ (that is, if $x, y \in P_E$ and $x \perp y$, then $x \oplus y \in P_E$) then

$$\forall_{x,y \in P_E} \quad x \perp y \Rightarrow x' \wedge (x \oplus y) = y.$$

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Proof.

Let $x, y \in P_E$ and $x \perp y$. Then $y \leq x'$ and $y \leq x \oplus y$ so y is a lower bound of x' and $x \oplus y$. Let t be a lower bound of x' and $x \oplus y$. We show that $t \leq y$. We know that $t \leq x'$ so $t \perp x$. Moreover $t \leq x \oplus y$ and $x \leq x \oplus y$ hence

$$x \oplus t \leq x \oplus y$$

since $x \oplus y \in P_E$. After using cancellation law we obtain $t \leq y$. Hence y is the largest lower bound of x' and $x \oplus y$ so $x' \wedge (x \oplus y) = y$. □

Effect algebras closed under $\oplus$-cont.

Lemma (8)

Let $(E, 0, 1, \oplus)$ *be an effect algebra. If for every* $x, y \in S_E$ *such that* $x \perp y$ *there exists* $x \vee y$ *in* E *and*

$$x' \wedge (x \vee y) = y, \quad x \oplus y = x \vee y \tag{9}$$

then S_E *is closed under* $\oplus$ *(that is, if* $x, y \in S_E$ *and* $x \perp y$*, then* $x \oplus y \in S_E$*).*

Effect algebras closed under $\oplus$-cont.

Lemma (8)

Let $(E, 0, 1, \oplus)$ be an effect algebra. If for every $x, y \in S_E$ such that $x \perp y$ there exists $x \vee y$ in E and

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then S_E is closed under $\oplus$ (that is, if $x, y \in S_E$ and $x \perp y$, then $x \oplus y \in S_E$).

Proof.

Let $x, y \in S_E$ and $x \perp y$. By (9) we have

$$0 = y' \wedge y = y' \wedge (x' \wedge (x \vee y)) \overset{Lemma\ (3)}{=} (x \vee y)' \wedge (x \vee y)$$

$$= (x \oplus y)' \wedge (x \oplus y),$$

Effect algebras closed under $\oplus$-main Theorem

In the following Theorem we show that if $S_E = P_E$ then $S_E = P_E$ is closed under $\oplus$ if and only if elements in $S_E = P_E$ satisfy the orthomodular law. It partially solves Open problems 3.2 and 3.3 in [9].

Theorem (9)

Let $(E, 0, 1, \oplus)$ be an effect algebra such that $S_E = P_E$. Then $S_E = P_E$ is closed under $\oplus$ if and only if for every $x, y \in S_E$ we have

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y.$$

Proof

Suppose that $S_E = P_E$ is closed under $\oplus$. Let $x, y \in S_E$ and $x \leq y$ then $x \perp y'$ and by Lemma (7) we have $x' \wedge (x \oplus y') = y'$ since $y' \in S_E$. It follows that

$$y \overset{Lemma\ (4)}{=} x \vee (x \oplus y')' \overset{Lemma\ (2)}{=} x \vee (x \vee y')'$$

$$\overset{Lemma\ (4)}{=} x \vee (x' \wedge y).$$

Proof-cont.

Suppose that for every $x, y \in S_E$ we have

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y \qquad (10).$$

We show that for every $x, y \in S_E$ such that $x \perp y$ there exists $x \vee y$ in E and

$$x' \wedge (x \vee y) = y, \quad x \oplus y = x \vee y.$$

Let $x, y \in S_E$ and $x \perp y$. Then $x \leq y'$ and $y' \in S_E = P_E$, so $x \vee (y')' = x \vee y$ exists in E and $x \vee y = x \oplus y$ by Lemma (2). Moreover $x \vee (x' \wedge y') = y'$ by (10). Hence $x' \wedge (x' \wedge y')' = y$ by Lemma (3) and $x' \wedge (x \vee y) = y$ by Lemma (4). Therefore S_E is closed under $\oplus$ by Lemma (8).

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