Special elements in bounded lattices with antitone involutions

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Introduction

- A lattice with antitone involutions is a bounded lattice $(L, \vee, \wedge, 0, 1)$ equipped with a collection $\{\neg_a : a \in L\}$ of antitone involutions (relative negations) $\neg_a$ on the principal filters $[a, 1] \subseteq L$.
- Examples:
  - Orthomodular lattices $(L, \vee, \wedge, ', 0, 1)$:

$$\neg_a x = x' \vee a \quad \text{for } x \geq a$$

  - MV-algebras $(A, \oplus, \neg, 0, 1)$:

$$\neg_a x = \neg x \oplus a \quad \text{for } x \geq a$$

  - Lattice effect algebras $(E, +, ', 0, 1)$:

$$\neg_a x = x' + a \quad \text{for } x \geq a$$

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- The so-called "basic algebras" ("AI-algebras"?) [Chajda-Halaš-Kühr 2009] are algebras $\boldsymbol{A} = (A, \oplus, \neg, 0, 1)$ corresponding to lattices with antitone involutions, as follows:
  - $\neg x = \neg_0 x$ and $x \oplus y = \neg_y(\neg x \vee y)$;
  - $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$, and $\neg_a x = \neg x \oplus a$ for $x \geq a$.
- For example, in orthomodular lattices:

$$x \oplus y = (x' \vee y)' \vee y = (x \wedge y') \vee y.$$

- The variety axiomatized by the identities:

$$x \oplus 0 = x,$$

$$\neg\neg x = x,$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$

$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1.$$

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Two boolean skeletons

Central elements

- An element $a \in A$ is central if $a = h(0,1)$ or $a = h(1,0)$ for some isomorphism $h\colon \boldsymbol{A}_1 \times \boldsymbol{A}_2 \cong \boldsymbol{A}$. Equivalently, $a \in \mathscr{C}(\boldsymbol{A})$ iff

$$\boldsymbol{\alpha}_a = \{(x,y) \in A^2 : x \vee a = y \vee a\}$$

is a factor congruence of $\boldsymbol{A} = (A, \oplus, \neg, 0, 1)$.
- $\mathscr{C}(\boldsymbol{A})$ is a boolean subalgebra of $\boldsymbol{A}$.

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Sharp elements

- An element $a \in A$ is sharp if $a \vee \neg a = 1$, or equivalently, if $a \oplus a = a$.
- In lattice effect algebras, $\mathcal{S}(\boldsymbol{A})$ is an orthomodular subalgebra.
- In basic algebras satisfying the identity

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z), \tag{M}$$

  $\mathcal{S}(\boldsymbol{A}) = \mathscr{C}(\boldsymbol{A})$.
- In general, $\mathcal{S}(\boldsymbol{A})$ is neither a subalgebra nor a sublattice. Is there a boolean subalgebra between $\mathscr{C}(\boldsymbol{A})$ and $\mathcal{S}(\boldsymbol{A})$?

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Boolean elements

- An element $a \in A$ is boolean if $a \oplus x = a \vee x$ for all $x \in A$. Note: If $a \in \mathscr{B}(\boldsymbol{A})$, then $\neg_x a = \neg a \vee x$ for every $x \leq a$.
- In lattice effect algebras, $\mathscr{B}(\boldsymbol{A}) = \mathscr{C}(\boldsymbol{A}) \subseteq \mathscr{S}(\boldsymbol{A})$.
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Distributive and standard elements

In any basic algebra $\boldsymbol{A}$, for any $a \in A$ one has:

- $a \in \mathrm{Distr}(\boldsymbol{A})$ iff $\neg a \in \mathrm{Distr}^{\partial}(\boldsymbol{A})$ iff for all $x, y \in A$,

$$(x \vee y) \oplus a = (x \oplus a) \vee (y \oplus a);$$

- $a \in \mathrm{Stand}(\boldsymbol{A})$ iff $\neg a \in \mathrm{Stand}^{\partial}(\boldsymbol{A})$ iff for all $x, y \in A$,

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Hence $\boldsymbol{A}$ is distributive iff it satisfies the identity

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If $\boldsymbol{A}$ is a lattice effect algebra, then

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"Strongly distributive" elements

Let $\boldsymbol{A} = (A, \oplus, \neg, 0, 1)$ be a basic algebra. The following are equivalent, for any $a \in A$:

- $h_a \colon x \mapsto x \vee a$ is a homomorphism of $\boldsymbol{A}$ onto the interval algebra $([a,1], \oplus_a, \neg_a, a, 1)$ where

  $$\neg_a x = \neg x \oplus a \quad \text{and} \quad x \oplus_a y = \neg(\neg x \oplus a) \oplus y$$

  for $x, y \geq a$;
- $\boldsymbol{\alpha}_a = \{(x,y) \in A^2 : x \vee a = y \vee a\}$ is a congruence of $\boldsymbol{A}$;
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- $x \oplus (y \vee a) = (x \oplus y) \vee a$ for all $x, y \in A$;
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In this case, $a \in \mathscr{B}(\boldsymbol{A})$.

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In this case, $a \in \mathcal{B}(\boldsymbol{A})$.

Central elements again

Let $\boldsymbol{A}$ be a basic algebra. The following are equivalent, for any $a \in A$:

- $a \in \mathscr{C}(\boldsymbol{A})$;
- $x \oplus (y \vee z) = (x \oplus y) \vee z$ for all $x, y \in A$ and $z \in \{a, \neg a\}$;
- $a \in \mathcal{S}(\boldsymbol{A})$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y \in A$ and $z \in \{a, \neg a\}$.

Boolean elements again

- If $a \in \mathscr{B}(\boldsymbol{A})$, then

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is a congruence of $(A, \vee, \wedge, \neg, 0, 1)$, not of $\boldsymbol{A}$ if $a \notin \mathscr{C}(\boldsymbol{A})$. If I is an ideal of the boolean algebra $\mathscr{B}(\boldsymbol{A})$ or $\mathscr{C}(\boldsymbol{A})$, then

$$\boldsymbol{\alpha}_I = \bigcup\{\boldsymbol{\alpha}_a : a \in I\}$$

is a congruence of $(A, \vee, \wedge, \neg, 0, 1)$ or $\boldsymbol{A}$, respectively.

- For any basic algebra $\boldsymbol{A}$,
  - $(A, \vee, \wedge, \neg, 0, 1)$ is a subdirect product of the quotient algebras $(A, \vee, \wedge, \neg, 0, 1)/\boldsymbol{\alpha}_M$ where M ranges over the maximal ideals of $\mathscr{B}(\boldsymbol{A})$;
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Pseudocomplements

- By a pseudocomplemented basic algebra we mean a pair $(\boldsymbol{A}, ^*)$ where $\boldsymbol{A} = (A, \oplus, \neg, 0, 1)$ is a basic algebra and * is a pseudocomplementation on it.
- In fact, $(A, \vee, \wedge, ^*, ^+, 0, 1)$, where

$$x^+ = \neg(\neg x)^*,$$

  is a double p-algebra.
- The variety of pseudocomplemented basic algebras is axiomatized by the axioms of basic algebras together with $0^* = 1$, $1^* = 0$ and $x \wedge (x \wedge y)^* = x \wedge y^*$.

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Pseudocomplements

- For any pseudocomplemented basic algebra $(\boldsymbol{A}, {}^*)$,

$$\mathcal{S}(\boldsymbol{A}) \cap \mathrm{Distr}(\boldsymbol{A}) \subseteq A^* = \{x^* : x \in A\},$$
$$\mathcal{S}(\boldsymbol{A}) \cap \mathrm{Distr}^\partial(\boldsymbol{A}) \subseteq A^+ = \{x^+ : x \in A\}.$$

  For $a \in \mathcal{S}(\boldsymbol{A})$, $a^+ \leq \neg a \leq a^*$.
- $\boldsymbol{A}$ is distributive iff it is modular, in which case $\mathcal{S}(\boldsymbol{A}) \subseteq A^* \cap A^+$ and $\neg a = a^* = a^+$ for $a \in \mathcal{S}(\boldsymbol{A})$.

Pseudocomplements

- Let $(\boldsymbol{A}, {}^*)$ be a pseudocomplemented basic algebra such that $A^* \subseteq \mathcal{S}(\boldsymbol{A})$, or equivalently, $A^+ \subseteq \mathcal{S}(\boldsymbol{A})$. Then $(\boldsymbol{A}, {}^*)$ satisfies the Stone identities

$$x^{**} \vee x^* = 1 \quad \text{and} \quad x^{++} \wedge x^+ = 0.$$

- If $(\boldsymbol{A}, {}^*)$ satisfies the identity

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then $\mathscr{C}(\boldsymbol{A}) = \mathscr{B}(\boldsymbol{A}) = \mathcal{S}(\boldsymbol{A}) = A^* = A^+$ and the double p-algebra $(A, \vee, \wedge, {}^*, {}^+, 0, 1)$ is a double Stone algebra.

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- We have $\mathscr{C}(\boldsymbol{A}) = A^*$ iff $(\boldsymbol{A}, {}^*)$ satisfies the equations

$$x \oplus (y \vee z^*) = (x \oplus y) \vee z^*,$$
$$x \oplus (y \vee \neg z^*) = (x \oplus y) \vee \neg z^*.$$

- Let $(\boldsymbol{A}, {}^*)$ be a pseudocomplemented basic algebra such that $\mathscr{C}(\boldsymbol{A}) = A^*$. Then the following are equivalent:
  - $(\boldsymbol{A}, {}^*)$ is subdirectly irreducible;
  - $\boldsymbol{A}$ is linearly ordered;
  - $(\boldsymbol{A}, {}^*)$ is simple.
- Pseudocomplemented basic algebras $(\boldsymbol{A}, {}^*)$ satisfying $\mathscr{C}(\boldsymbol{A}) = A^*$ form a discriminator variety:

$$t(x, y, z) = (d(x, y)^* \vee x) \wedge (d(x, y)^{**} \vee z)$$

where $d(x, y) = \neg(\neg x \oplus y) \vee \neg(\neg y \oplus x)$.

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Pseudocomplements

Corollary. Every finite basic algebra satisfying

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z) \tag{M}$$

is an MV-algebra.

Thank you!