Free skew Boolean algebras

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Plan of the talk

1. Preliminaries on skew Boolean algebras.
2. Background: free generalized Boolean algebras.
3. Free skew Boolean algebras.
4. Free skew Boolean intersection algebras.

Generalized Boolean algebras

A generalized Boolean algebra (GBA) is an algebra $(A; \wedge, \vee, \setminus, 0)$ of signature $(2, 2, 2, 0)$ satisfying the following axioms:

1. (associativity) $a \vee (b \vee c) = (a \vee b) \vee c$,
   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
2. (commutativity) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;
3. (absorption) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$;
4. (distributivity) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
5. (properties of 0) $a \vee 0 = a$, $a \wedge 0 = 0$;
6. (properties of relative complement) $(a \setminus b) \wedge b = 0$,
   $(a \setminus b) \vee (a \wedge b) = a$.

Thus GBAs form a variety of algebras.
Associativity, commutativity and absorption imply

7. (idempotency) $a \vee a = a$, $a \wedge a = a$.

Skew Boolean algebras (SBAs)

A skew Boolean algebra is an algebra $(S; \wedge, \vee, \setminus, 0)$ of type $(2, 2, 2, 0)$ such that for any $a, b, c, d \in S$:

1. (associativity) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
2. (absorption) $a \vee (a \wedge b) = a$, $(b \wedge a) \vee a = a$, $a \wedge (a \vee b) = a$, $(b \vee a) \wedge a = a$;
3. (distributivity) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a)$;
4. (properties of 0) $a \vee 0 = 0 \vee a = a$, $a \wedge 0 = 0 \wedge a = 0$;
5. (properties of relative complement) $(a \setminus b) \wedge b = b \wedge (a \setminus b) = 0$, $(a \setminus b) \vee (a \wedge b \wedge a) = a = (a \wedge b \wedge a) \vee (a \setminus b)$;
6. (normality) $a \wedge b \wedge c \wedge d = a \wedge c \wedge b \wedge d$.

Associativity and absorption imply that $(A, \wedge, \vee)$ is a *skew lattice* and the absorption axiom implies

7. (idempotency) $a \vee a = a$, $a \wedge a = a$.

Skew Boolean intersection algebras (SBIAs)

Let $(S; \wedge, \vee, \setminus, 0)$ be an SBA.

- The natural partial order on S: $a \leq b$ iff $a \wedge b = b \wedge a = a$.
- S has intersections if the meet of a and b with respect to $\leq$ exists for any $a, b \in S$.
- $a \sqcap b$ — the intersection of a and b.
- If $(S; \wedge, \vee, \setminus, 0)$ has intersections, $(S; \wedge, \vee, \sqcap, \setminus, 0)$ is a skew Boolean intersection algebra (SBIA).

Proposition (Bignall and Leech)

Let $(S; \wedge, \vee, \setminus, \sqcap, 0)$ be an algebra of type $(2, 2, 2, 2, 0)$. Then it is an SBIA if and only if $(S; \wedge, \vee, \setminus, 0)$ is an SBA, $(S, \sqcap)$ is a semilattice (meaning that $\sqcap$ is idempotent and commutative) and the following identities hold:

$$x \sqcap (x \wedge y \wedge x) = x \wedge y \wedge x; \quad x \wedge (x \sqcap y) = x \sqcap y = (x \sqcap y) \wedge x.$$

Historical note

- 1949 - Pascual Jordan: work on non-commutative lattices.
- early 1970's - a series of works by Boris Schein
- late 1970's and early 1980's - Robert Bignall and William Cornish studied non-commutative Boolean algebras
- 1989 - Jonathan Leech initiated the modern study of skew lattices.
- 1995 - Jonathan Leech and Robert Bignall publish the first paper devoted to skew Boolean intersection algebras.
- 1989 - present many aspects of skew lattices and skew Boolean algebras have been studied.
- 2012-2013 - Stone duality was extended to skew Boolean algebras and distributive skew lattices.
- A. Bauer, K. Cvetko-Vah, M. Gehrke, S. van Gool, M. Kinyon, GK, M. V. Lawson, J. Leech, J. Pita-Costa, M.Spinks is an (incomplete) list of researchers who have contributed to the topic.

Left-handed SBAs

An SBA S is left-handed if the normality axiom is replaced by:

(6′) (left normality) $x \wedge y \wedge z = x \wedge z \wedge y$,

and a dual axiom holds for right-handed SBAs.

In this talk, we restrict attention to left-handed SBAs. The right-handed case is dual, and the general case can be easily obtained from these two applying Leech's fibered product decomposition theorem.

Notation:

- $LSBA$ - left-handed SBA.
- $LSBIA$ - left-handed SBIA.

On the structure of LSBAs and LSBIAs

- Relation $\mathcal{D}$: $x \, \mathcal{D} \, y$ if and only if $x \wedge y = x$ and $y \wedge x = y$.
- If S is an SBA then $\mathcal{D}$ is a congruence on S and $S/\mathcal{D}$ is the maximum commutative quotient of S. (But, if S is an SBIA, $\mathcal{D}$ is in general not respected by $\sqcap$ operation.)
- Primitive LSBAs: $(\mathbf{k}+\mathbf{1})_L = \{0, \ldots, k\}$ $i \wedge j = i$, $i \vee j = j$ for any $1 \leq i, j \leq k$. This generalizes the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$.
- Fact: Let S be a finite LSBIA. Then

$$S \simeq \mathbf{2}^{k_2} \times \mathbf{3}_L^{k_3} \times \cdots \times (\mathbf{m}+\mathbf{1})_L^{k_{m+1}}$$

  for some m where all $k_i \geq 0$. This generalizes the fact that any finite Boolean algebra is isomorphic to some $\mathbf{2}^k$.

Why are LSBAs important and worth attention?

1. Partial functions algebras are LSBAs generalizing powersets and are prototypical examples of finite SBAs.
   $\mathcal{P}(A, \{1, \ldots, k\}) \simeq (\mathbf{k}+\mathbf{1})_L^{|A|}$ similarly as
   $\mathcal{P}(A) \simeq \mathcal{P}(A, \{1\}) \simeq \mathbf{2}^{|A|}$.
2. LSBAs arise naturally in rings where idempotents are closed under multiplication (Cvetko-Vah and Leech).
3. The category of LSBAs is dual to the category of étale spaces over Boolean spaces. (GK, 2012).
4. The category of LSBIAs is dual to the category of Hausdorff étale spaces over Boolean spaces (Bauer and Cvetko-Vah, GK, 2012).

Free generalized Boolean algebras

The free generalized Boolean algebra $\mathbf{GBA}_X$ over the generating set X is the the algebra of all terms over X, where two terms are equal in $\mathbf{GBA}_X$ if one of them can be obtained from another one by a finite number of applications of the identities defining the variety of GBAs.

Example. Let $X = \{x_1, x_2\}$. Then $\mathbf{GBA}_X \simeq \mathbf{2}^3$ where $\mathbf{2} = \{0, 1\}$. Indeed, denote $a_{\{1\}} = x_1 \setminus x_2$, $a_{\{2\}} = x_2 \setminus x_1$ and $a_{\{1,2\}} = x_1 \wedge x_2$.

- These terms are pairwise distinct, as they can have distinct evaluation in $\mathbf{2}$.
- $a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3 = 0$.
- $x_1 = (x_1 \setminus x_2) \vee (x_1 \wedge x_2)$ and $x_2 = (x_2 \setminus x_1) \vee (x_1 \wedge x_2)$.
- Thus any element of $\mathbf{GBA}_X$ is a join of a subset of $\{a_1, a_2, a_3\}$.

Similarly, if $|X| = n$, $\mathbf{GBA}_X \simeq \mathbf{2}^{2^n-1}$, atoms are in a bijections with non-empty subsets of X.

Free LSBAs: finite case

${}_{\mathcal{L}}\mathbf{SBA}_X$ — the free LSBA over the generating set X. Let $X = \{1, 2, \ldots, n\}$. Can we describe atoms and atomic $\mathcal{D}$-classes of ${}_{\mathcal{L}}\mathbf{SBA}_X$?

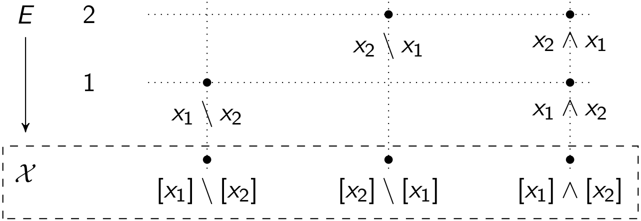
- Let $Y \subseteq X$ be a non-empty subset and $y \in Y$.
- (X, Y, y) – pointed non-empty subset of X.
- Atoms:

$$e(X, Y, y) = (y \wedge (\wedge\{y \colon y \in Y))) \setminus \vee\{y \colon y \in X \setminus Y\}.$$

  By left normality, this is well defined and $e(X, Y, y) = e(X, Z, z)$ if and only if $Y = Z$ and $y = z$.
- Atoms are in a bijection with pointed non-empty subsets of X.
- Atomic $\mathcal{D}$-classes are in a bijection with non-empty subsets of X.

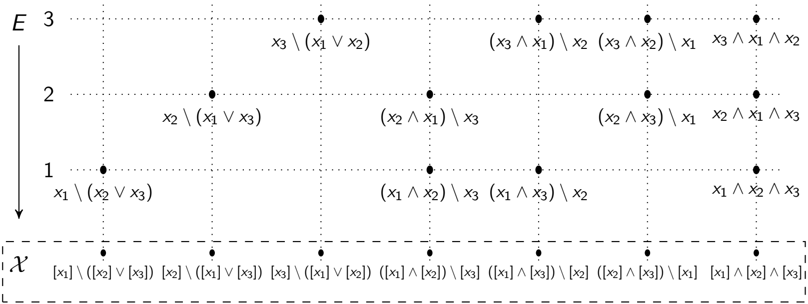
Example

Atoms of ${}_{\mathcal{L}}\mathbf{SBA}_2$:



Example

Atoms of ${}_{\mathcal{L}}\mathbf{SBA}_3$:



Free LSBAs: finite case, structure

Theorem (Jonathan Leech and GK, 2015)

Let $n \geq 1$.

1. $$_{\mathcal{L}}\mathbf{SBA}_n \simeq \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_L^{\binom{n}{2}} \times \mathbf{4}_L^{\binom{n}{3}} \times \cdots \times (\mathbf{n+1})_L^{\binom{n}{n}}.$$

   Consequently,

   $$|_{\mathcal{L}}\mathbf{SBA}_n| = 2^{\binom{n}{1}} 3^{\binom{n}{2}} 4^{\binom{n}{3}} \ldots (n+1)^{\binom{n}{n}}.$$

2. *The number of atoms of $_{\mathcal{L}}\mathbf{SBA}_n$ equals*

   $$\binom{n}{1}1 + \binom{n}{2}2 + \cdots + \binom{n}{n-1}(n-1) + \binom{n}{n}n = n2^{n-1}.$$

3. *The center of $_{\mathcal{L}}\mathbf{SBA}_n$ is isomorphic to $\mathbf{2}^n$.*

Free LSBAs: infinite case

Theorem (Jonathan Leech and GK, 2015)

Let S be a LSBA, let $X \subseteq S$ be a generating set of S and let $\pi\colon S \to S/\mathcal{D}$ be the canonical homomorphism. TFAE:

(i) *S is freely generated by X.*

(ii) *For every finite $Y \subseteq X$, the subalgebra $\langle Y \rangle$ is free on Y.*

(iii) *For every subset $\{x_1, \ldots, x_n\}$ of n distinct elements in X, their evaluations in the $n2^{n-1}$ atomic terms on n variables produce $n2^{n-1}$ distinct non-zero outcomes in S.*

(iv) *$S/\mathcal{D}$ is freely generated by $\pi(X)$ and for any $x \neq y \in X$, $x \cap y$ exists and equals 0. Thus ${}_{\mathcal{L}}\mathbf{SBA}_X$ has intersections.*

Proposition

1. *${}_{\mathcal{L}}\mathbf{SBA}_X$ is atomless.*
2. *The center of ${}_{\mathcal{L}}\mathbf{SBA}_X$ equals $\{0\}$.*

Free LSBAs: infinite case, continued

- Let X be infinite and put $\mathcal{X} = \{0,1\}^X \setminus \{f_0\}$ where $f_0 = 0$.
- $\Omega = \{(f,x)\colon f \in \mathcal{X}$ and $x \in X$ is such that $f(x) = 1\}$.
- $p\colon \Omega \to \mathcal{X}\colon\ p(f,x) = f$, $\mathbf{S}_X = \{A \subseteq \Omega\colon\ p|_A$ is injective.$\}$
- Define the binary operations $\vee$, $\wedge$ and $\setminus$ on $\mathbf{S}_X$ by:

$$\begin{array}{rcl} A \wedge B & = & \{(f,x) \in A\colon f \in p(A) \cap p(B)\}, \\ A \vee B & = & (A \setminus B) \cup B, \\ A \setminus B & = & \{(f,x) \in A\colon f \in p(A) \setminus p(B)\}. \end{array}$$

- $(\mathbf{S}_X; \wedge, \vee \setminus, \varnothing)$ is a LSBIA.
- $i\colon X \to \mathbf{S}_X$, $i(x) = \{(f,x)\colon f(x) = 1\}$, $\overline{X} = \{i(x)\colon x \in X\}$, $\mathbf{S}_X = \langle \overline{X} \rangle$.
- The evaluation of $e(Z, Y, y)$, where $Z \subseteq X$ is finite, is $\{(f,y) \in \Omega\colon f(x) = 1, x \in Y$ and $f(x) = 0, x \in Z \setminus Y\}$.
- $\mathbf{S}_X$ is freely generated by $\overline{X}$.

Free LSBIAs: finite case

${}_{\mathcal{L}}\mathbf{SBIA}_X$ — the free LSBIA over the generating set X.

- free LSBAs: atoms encoded pointed non-empty subsets of X.
- free LSBIAs: atoms encoded by pointed partitions of non-empty subsets of X!

The construction (GK, 2016)

Let (X, α, A) be a pointed partition of a non-empty subset of X. Let $\alpha = \{A_1, \ldots, A_k\}$ (thus $A = A_i$ for some i) and $Y = \mathrm{dom}(\alpha)$. Define

$$e(X, \alpha, A) = p \setminus (\vee Q), \text{ where}$$

$$p = (\sqcap A) \wedge (\wedge\{\sqcap A_i \colon 1 \leq i \leq k\}) = (\sqcap A) \wedge (\sqcap A_1) \wedge \cdots \wedge (\sqcap A_k) \text{ and}$$

$$Q = (X \setminus Y) \cup \{\sqcap(A_i \cup A_j) \colon 1 \leq i < j \leq k\}.$$

Free LSBIAs: finite case continued

Example. *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$, $\alpha = x_1x_3|x_4$, $\beta = x_2|x_3x_4|x_5$ *and* $\gamma = x_1x_2x_3x_4x_5$. *Then:*

$$e(X, \alpha, \{x_4\}) = (x_4 \wedge (x_1 \sqcap x_3)) \setminus (x_2 \vee x_5 \vee (x_1 \sqcap x_3 \sqcap x_4)),$$

$$e(X, \alpha, \{x_1, x_3\}) = ((x_1 \sqcap x_3) \wedge x_4) \setminus (x_2 \vee x_5 \vee (x_1 \sqcap x_3 \sqcap x_4)),$$

$$e(X, \beta, \{x_3, x_4\}) =$$
$$((x_3 \sqcap x_4) \wedge x_2 \wedge x_5) \setminus (x_1 \vee (x_2 \sqcap x_3 \sqcap x_4) \vee (x_2 \sqcap x_5) \vee (x_3 \sqcap x_4 \sqcap x_5)),$$

$$e(X, \gamma, \{x_1, x_2, x_3, x_4, x_5\}) = x_1 \sqcap x_2 \sqcap x_3 \sqcap x_4 \sqcap x_5.$$

Free LSBIAs: finite case continued

Atoms

The elements $e(X, \alpha, A)$ are all pairwise distinct and non-zero, and they are precisely the atoms of ${}_{\mathcal{L}}\mathbf{SBIA}_X$. Thus atoms are in a bijection with pointed partitions of non-empty subsets of X.

The relation of containment of partitions

Let $Z \subseteq Y$, $Z \neq \varnothing$, and α, β be partitions of non-empty subsets of Z and Y, respectively. Define $(Z, \alpha) \preceq (Y, \beta)$ if

$$\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\beta) \subseteq \mathrm{dom}(\alpha) \cup (Y \setminus Z)$$

and for any $x, y \in \mathrm{dom}(\alpha)$: $x \; \alpha \; y$ if and only if $x \; \beta \; y$.

Example

Let $Y = \{x_1, x_2, x_3, x_4\}$, $Z = \{x_1, x_2, x_3\}$, $\alpha = x_1|x_2$, $\beta = x_1|x_2x_4$ $\gamma = x_1x_3|x_2x_4$. Then $(Z, \alpha) \preceq (Y, \beta)$, but $(Z, \alpha) \not\preceq (Y, \gamma)$ since $\mathrm{dom}(\gamma) \not\subseteq \mathrm{dom}(\alpha) \cup (Y \setminus Z)$.

Free LSBIAs: finite case continued

If (X, α, A) is a pointed partition and $(X, \alpha) \preceq (Y, \beta)$ then there is the only block of β which contains A, denoted $A{\uparrow}_{\alpha}^{\beta}$.

Theorem (Decomposition Rule)

Let S be a LSBIA, X, Y be finite non-empty subsets of S and $X \subseteq Y$. Let (X, α, A) be a pointed partition. Then

$$e(X, \alpha, A) = \vee\{e\left(Y, \beta, A{\uparrow}_{\alpha}^{\beta}\right) : (X, \alpha) \preceq (Y, \beta)\}, \qquad (1)$$

the latter join being orthogonal.

Free LSBIAs: finite case, structure

The nth *Bell number*, B_n, equals the number of partitions of an n-element set. The *Stirling number of the second kind*, $\left\{ n \atop k \right\}$, equals the number of partitions of an n-element set into k non-empty subsets, $n \geq 1$, $1 \leq k \leq n$.

Theorem (GK, 2016)

1. ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *has precisely* $B_{n+1} - 1$ *atomic* $\mathcal{D}$*-classes. Thus* ${}_{\mathcal{L}}\mathbf{SBIA}_n/\mathcal{D} \simeq \mathbf{2}^{B_{n+1}-1}$.
2. ${}_{\mathcal{L}}\mathbf{SBIA}_n \simeq \mathbf{2}^{\left\{ n+1 \atop 2 \right\}} \times \mathbf{3}_L^{\left\{ n+1 \atop 3 \right\}} \times \cdots \times (\mathbf{n+1})_L^{\left\{ n+1 \atop n+1 \right\}}$, $|{}_{\mathcal{L}}\mathbf{SBIA}_n| = 2^{\left\{ n+1 \atop 2 \right\}} 3^{\left\{ n+1 \atop 3 \right\}} \cdots (n+1)^{\left\{ n+1 \atop n+1 \right\}}$.
3. ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *has precisely* $B_{n+2} - 2B_{n+1}$ *atoms.*
4. *The center of* ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *is isomorphic to* $\mathbf{2}^{2^n-1}$ *which is isomorphic to* $\mathbf{GBA}_n$ *which is isomorphic to the maximum commutative quotient of* ${}_{\mathcal{L}}\mathbf{SBIA}_n$.

Free LSBIAs: infinite case

- $\mathcal{X} = \{(X, \alpha) \colon \alpha \in \mathcal{P}(Y)$ where $Y \subseteq X$ and $Y \neq \varnothing\}$.
- $\Omega = \{(X, \alpha, A) \colon (X, \alpha) \in \mathcal{X}$ and $A \in \alpha\}$.
- $p \colon \Omega \to \mathcal{X}$, $p(X, \alpha, A) = (X, \alpha)$.
- $\mathbf{S}_X = \{U \subseteq \Omega \colon p|_U$ is injective$\}$.
- On $\mathbf{S}_X$ we define the binary operations $\vee$, $\wedge$, $\setminus$ and $\sqcap$ by:

$$\begin{array}{rcl} U \wedge V & = & \{(X, \alpha, A) \in U \colon (X, \alpha) \in p(U) \cap p(V)\}, \\ U \vee V & = & (U \setminus V) \cup V, \\ U \setminus V & = & \{(X, \alpha, A) \in U \colon (X, \alpha) \in p(U) \setminus p(V)\}, \\ U \sqcap V & = & U \cap V. \end{array}$$

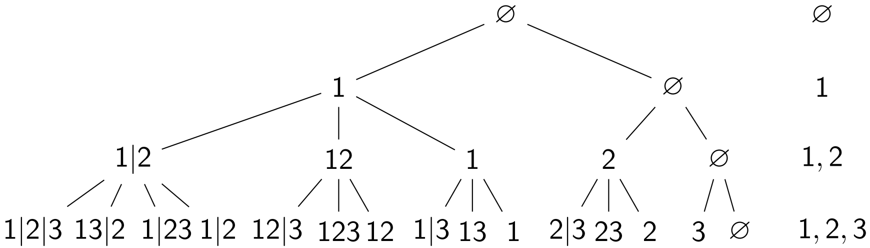
- $(\mathbf{S}_X; \wedge, \vee \setminus, \varnothing, \sqcap)$ is a LSBIA.
- $i \colon X \to \mathbf{S}_X$, $i(x) = \{(X, \alpha, A) \colon x \in \mathrm{dom}(\alpha)$ and $x \in A\}$.
- $\overline{X} = \{i(x) \colon x \in X\}$, $\mathbf{S}_X = \langle \overline{X} \rangle$.
- The evaluation of $e(X_n, \alpha, A)$ on $\{i(x_1), i(x_2), \ldots, i(x_n)\}$:

$$\{(X, \beta, A{\uparrow}_{\alpha}^{\beta}) \in \Omega \colon (X_n, \alpha) \preceq (X, \beta)\}.$$

- $\mathbf{S}_X$ is freely generated by $\overline{X}$.

Free LSBAs: countable generating set

Infinite partition tree: level i: partitions of subsets of $[i] = \{1, 2, \ldots, i\}$. $([i], \alpha)$ is connected with $([i+1], \beta)$ iff $([i], \alpha) \preceq ([i+1], \beta)$.



This tree is Cantorian, so that its boundary is homeomorphic to the Cantor set. Basis of topology: sets $[v]$ = all paths passing through v, where v ranges over the vertices. Thus ${}_{\mathcal{L}}\mathbf{SBIA}_{\mathbb{N}}/\mathcal{D} \simeq \mathbf{GBA}_{\mathbb{N}}$.

Main references

1. G. Kudryavtseva, J. Leech, Free skew Boolean algebras, preprint, arXiv:1510.07539.

2. G. Kudryavtseva, Free skew Boolean intersection algebras and set partitions, preprint, arXiv:1602.01789.