On Łukasiewicz congruences and subalgebras

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T-norms

Definition

A function $*: [0,1]^2 \longrightarrow [0,1]$ is called a *t-norm* if it satisfies the following properties:

- $(a * b) * c = a * (b * c)$
- $a * b = b * a$
- $1 * a = a$
- $a \leq b \Rightarrow a * c \leq b * c$

A t-norm is called *continuous* if it is continuous as a function.

- One of the possible bases for fuzzy equational logic

Continuous t-norms

For every continuous t-norm, there exists a unique residue $\rightarrow$ such that

$$a * b \leq c \iff a \leq b \rightarrow c.$$

Continuous t-norms can be constructed using basic t-norms:

- Gödel t-norm
  $a * b = \min\{a, b\}$
- Product t-norm
  $a * b = a \cdot b$
- Łukasiewicz t-norm
  $a * b = \max\{0, (a + b) - 1\}$

MV-algebras and Łukasiewicz t-norm

Definition

An *MV-algebra* $\mathcal{A} = (A, \oplus, \neg, 0)$ is an algebra of a type (2,1,0), where $(A, \oplus, 0)$ is a commutative monoid, and which satisfies the following identities:

- $\neg\neg a = a$
- $a \oplus \neg 0 = \neg 0$
- $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$

The derived operation $\odot$ is defined as $a \odot b = \neg(\neg a \oplus \neg b)$. Then Łukasiewicz t-norm is the derived operation $\odot$ of the *Standard MV-algebra*

$$\mathcal{S} = ([0,1], \oplus, \neg, 0)$$

where $a \oplus b = \min\{1, a + b\}$ and $\neg a = 1 - a$.

MV-subalgebras

Definition

Let $\mathcal{A} = (A, F)$ be an algebra. A function $\alpha : A \longrightarrow [0, 1]$ such that

- $\alpha(c) = 1$ for every $c \in F_0$
- $\bigodot_{i=1}^{n} \alpha(a_i) \leq \alpha(f(a_1, \ldots, a_n))$ for every $f \in F_n$ and $a_1, \ldots, a_n \in A$

is called an *MV-subalgebra* of $\mathcal{A}$.

Łukasiewicz t-norm is the only one of the basic t-norms such that derived ”subalgebras” are closed under:

- convex combinations
- limits

Definition

Let $\mathcal{A} = (A, F)$ be an algebra. A function $\theta : A^2 \longrightarrow [0, 1]$ is called an *MV-congruence* on $\mathcal{A}$ if it satisfies the following:

- $\theta(a, a) = 1$
- $\theta(a, b) = \theta(b, a)$
- $\theta(a, b) \odot \theta(b, c) \leq \theta(a, c)$
- $\bigodot_{i=1}^{n} \theta(a_i, b_i) \leq \theta(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)),\ f \in F_n$

There is a one-to-one correspondence between MV-congruences on $\mathcal{A} = (A, F)$ and MV-subalgebras of $\mathcal{A}' = (A^2, F')$ such that $F' = F \cup \{(a, a) \mid a \in A\} \cup \{s, t\}$ where

- $s((a, b)) = (b, a)$
- $t((a_1, b_1), (a_2, b_2)) = \begin{cases} (a_1, b_2) & \text{if } b_1 = a_2 \\ (a_1, b_1) & \text{otherwise} \end{cases}$

Let α_i be an MV-subalgebra for $i \in I$, than a function α such that

$$\alpha(x) = \bigwedge_{i \in I} \alpha_i(x)$$

is an MV-subalgebra.
Let $S_{MV}(\mathcal{A})$ denote the set of all MV-subalgebras of $\mathcal{A}$.

Definition

For $\gamma : A \longrightarrow [0,1]$ we define $[\gamma]$ as the smallest MV-subalgebra greater than α, i.e.,

$$[\gamma] = \bigwedge \{\alpha_i \in S_{MV}(\mathcal{A}) \mid \gamma \leq \alpha_i\}$$

For any MV-subalgebra α of $\mathcal{A}$ we may define its *levels* as

$$L_k = \{a \in A \mid \alpha(a) \geq k\}$$

where $k \in [0, 1]$ is fixed.

- L_1 is a subalgebra of $\mathcal{A}$
- If $t(y_1, \ldots, y_n, \underline{x})$ is a term of the language F such that x has only one instance in t (denoted as $\underline{x}$) and $a_1, \ldots, a_n \in L_1$, $b \in L_k$ than
$$t(a_1, \ldots, a_n, \underline{b}) \in L_k.$$

Definition

A subset $L \subseteq A$ is called a *level over the subalgebra S* if for every term $t(y_1, \ldots, y_n, \underline{x})$ of the language F and $a_1, \ldots, a_n \in S$, $b \in L$ the condition

$$t(a_1, \ldots, a_n, \underline{b}) \in L.$$

holds.

Let L be a level over the subalgebra S. We define a function $d_L : A \longrightarrow \mathbb{N} \cup \{\infty\}$ as follows

$$d_L(x) = n$$

where n is the smallest number such that there exists a term $t(y_1, \ldots, y_m, \underline{x_1, \ldots, x_n})$ and elements $a_1, \ldots, a_m \in S$, $b_1, \ldots, b_n \in L$ for which

$$x = t(a_1, \ldots, a_m, \underline{b_1, \ldots, b_n}).$$

Theorem

Let L_k be levels of an MV-subalgebra α than

$$\alpha(x) = \bigvee_{k \in [0,1]} k^{d_{L_k}(x)}$$

where the power is implemented using Łukasiewicz t-norm.

Definition

A function $f : [0,1] \longrightarrow [0,1]$ is called a $\odot$-*function* if it satisfies the following conditions

- $g(1) = 1$
- $a \leq b \Longrightarrow f(a) \leq f(b)$
- $f(a) \odot f(b) \leq f(a \cdot b)$

Let α be an MV-subalgebra and f a $\odot$-function than

$$\beta = \alpha \circ f$$

is also an MV-subalgebra.

Super-additive functions

Definition

A function $f : [0,1] \longrightarrow [0,1]$ is called a $\oplus$-*function* if it satisfies the following conditions

- $f(0) = 0$
- $a \leq b \Longrightarrow f(a) \leq f(b)$
- $f(a+b) \leq f(a) \oplus f(b)$

The set of $\oplus$-functions (denoted *SA*) is closed under limits, convex combinations and suprema.

Theorem

The set SA can be generated by the set of all piecewise constant $\oplus$*-functions.*

Let $W = \{(a, b) \in [0, 1]^2 \mid a \leq b\}$. For $R = (C, S) \in \mathcal{P}(W)^2$ we define a set $F_R = F_C \cap F_S$ where

$$\begin{aligned} F_C &= \{f \in SA \mid f(a) = f(b) \quad \forall (a, b) \in C\} \\ F_S &= \{f \in SA \mid f(a) + f(b) = f(a + b) \quad \forall (a, b) \in S\}. \end{aligned}$$

We may also define a subset $R_F = (C_F, S_F) \in \mathcal{P}(W)^2$ of all pairs where identities above hold for every $f \in F \subseteq SA$. This would form a Galois connection.

Theorem

Let $R \in \mathcal{P}(W)^2$. If $F_R = \{f\}$ then f is an extreme in SA.

Finite patologies

To obtain finite $\oplus$-functions the domain of $\oplus$-functions can be restricted to

$$K_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\} \subseteq [0, 1].$$

The second implication of the previous theorem holds for the finite case, thus this case may seem more simple.
On the contrary:

- The staircase
- The ski-jupm
- The comfort ski-jupm
- The patologies

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Thank you for your attention!