Bounds on the length of terms in various types of algebras

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Terms

Definition

Let **A** be an algebra on the language $\mathcal{L}$. For each $n \in \mathbb{N}$ and variables $x_1, \ldots, x_n$ we define the set of all terms with variables $x_1, \ldots, x_n$ in abbreviation $T(x_1, \ldots, x_n)$ as the smallest set with

1. $x_i \in T(x_1, \ldots, x_n)$ for each $i \in \{1, \ldots, n\}$;
2. if $t_1, \ldots, t_k \in T(x_1, \ldots, x_n)$ and $f \in \mathcal{L}$ of the arity $k \in \mathbb{N}$ then $f(t_1, \ldots, t_k) \in T(x_1, \ldots, x_n)$.

Example

In the group $(\mathbb{Z}, +)$ we have $(x + y) + z \in T(x, y, z)$.

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Term operations

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Let $n \in \mathbb{N}$. With each term $t(x_1, \ldots, x_n)$ in the language of the algebra **A** we associate an n-ary term operation by interpretation of each operation symbol with corresponding operation in **A**. The set of all n-ary term functions of **A** we denote by $\mathrm{Clo}_n(\mathbf{A})$.

Remark

Sometimes we say circuit instead of term function.

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2. if $k \in \mathbb{N}$, $f \in \mathcal{L}$ and $t_1, \ldots, t_k \in T(x_1, \ldots, x_n)$ then $||f(t_1, \ldots, t_k)|| = 1 + ||t_1|| + \cdots + ||t_k||$.

Example

$||(x+y)+z|| = 1+||x+y||+||z|| = 1+(1+||x||+||y||)+||z|| = 5.$

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Term operations versus terms

Number of functions in finite algebras

If **A** is a finite algebra and $n \in \mathbb{N}$ then there is a finite number of distinct n-ary functions on the underlying set.

Remark

For each $n \in \mathbb{N}$, $T(x_1, \ldots, x_n)$ is an infinite set.

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Infinitely many different terms in $T(x_1, \ldots, x_n)$ represent the same n-ary term function, but we need only finitely many of them to represent all distinct n-ary term functions of the given algebra **A**.

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Circuit complexity problem

How to check?

Let $n \in \mathbb{N}$. Give an n-ary function on a finite algebra. Is it a term function (a circuit)?

When to stop?

A computer program can check all the n-ary terms starting from the smallest length, but when to stop? We should know the minimal length of terms such that all distinct term functions can be represented by terms of the length at most n.

The minimal length of terms

$$\gamma_{\mathbf{A}}(n) := \min\{m \in \mathbb{N} \mid (\forall f \in \mathrm{Clo}_n\mathbf{A})(\exists \mathtt{t})(||\mathtt{t}|| \leq m \wedge \mathtt{t}^{\mathbf{A}} = f)\}$$

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The task

What we want to do?

Let $n \in \mathbb{N}$. Find $\gamma_{\mathbf{A}}(n)$ for different types of algebra **A**.

Definition

Two algebras **A** and **B** are 2-equivalent if there exist polynomials p and q such that $\gamma_{\mathbf{A}}(n) \leq p(\gamma_{\mathbf{B}}(n))$ and $\gamma_{\mathbf{B}}(n) \leq q(\gamma_{\mathbf{A}}(n))$ for all $n \in \mathbb{N}$.

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The depth of terms

Definition

Let $\mathcal{L}$ be a language and let $n \in \mathbb{N}$. The depth of terms is a function $d : T(x_1, \ldots, x_n) \to \mathbb{N}$ defined by:

- $d(x_1) = \cdots = d(x_n) = 1$;
- $d(f(t_1, \ldots, t_k)) = max\{d(t_1), \ldots, d(t_k)\} + 1$ for all $f \in \mathcal{L}$, $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in T(x_1, \ldots, x_n)$.

Example

$d(x + (y + (z + t))) = 4.$

Notation

We denote the set of all terms in the language of an algebra **A**, over a set of variables X, of depth $n \in \mathbb{N}$ by $\delta_{\mathbf{A}}(X, n)$.

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Two observations on the depth

Lemma

Let **A** be an algebra, let X be a finite set of variables and let $k, n \in \mathbb{N}$, where $|X| = n$. If

$$\{t^{\mathbf{A}} \mid t \in \delta_{\mathbf{A}}(X, k+1)\} \subseteq \cup_{i=1}^{k} \{t^{\mathbf{A}} \mid t \in \delta_{\mathbf{A}}(X, i)\}$$

then $\mathrm{Clo}_n \mathbf{A} = \cup_{i=1}^{k} \{t^{\mathbf{A}} \mid t \in \delta_{\mathbf{A}}(X, i)\}$.

Lemma

Let $\mathbf{A} = (A, F)$ be an algebra, let X be a finite set of variables and let $k, m \in \mathbb{N}$. If $t \in \delta_{\mathbf{A}}(X, k)$ and $m = max\{ar(f) \mid f \in F\}$ then $||t|| \leq 1 + m + \cdots + m^{k-1}$.

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Upper and lower bound

Theorem (Upper bound)

Let $\mathbf{A} = (A, F)$ be an algebra, let $n \in \mathbb{N}$ and let $m = max\{ar(f) \mid f \in F\}$. Then $\gamma_{\mathbf{A}}(n) \leq 1 + m + \cdots + m^{|\mathrm{Clo}_n \mathbf{A}|-1}$.

Proposition (Lower bound)

Let $\mathbf{A} = (A, F)$ be an algebra, let $n \in \mathbb{N}$ and let $|F| = m, m \in \mathbb{N}$. If $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n)$ is the free algebra in the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ then, $\gamma_{\mathbf{A}}(n) \geq \log_{(m+n+2)} |\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)|$.

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Functionally complete algebras

Definition

Functionally complete algebra is an algebra such that each function is a polynomial function.

Theorem (ad hoc criteria)

Every functionally complete algebra is polynomially equivalent to an algebra **A** that has binary operations $+$ and $\cdot$ as fundamental operations as well as characteristic functions χ_a for all $a \in A$ such that $x + 0 = 0 + x = x$, $x \cdot 1 = x$ and $x \cdot 0 = 0$ for all $x \in A$.

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An upper bound

Theorem (E. Aichinger and N.M.)

Let $\mathbf{A}$ be a finite functionally complete algebra that has binary operations $+$ and $\cdot$ as fundamental operations as well as characteristic functions χ_a for all $a \in A$ such that $x + 0 = 0 + x = x$, $x \cdot 1 = x$ and $x \cdot 0 = 0$ for all $x \in A$. If $n \in \mathbb{N}$ then $\gamma_{\mathbf{A}}(n) = 3|A|^n(n+1) - 1$.

Theorem (E. Aichinger and N.M.)

Let $\mathbf{A} = (A, F)$ be a functionally complete algebra and let $n \in \mathbb{N}$. Then there is a $c \in \mathbb{N}$ such that $\gamma_{\mathbf{A}}(n) \leq |A|^{cn}$.

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Around groups

Proposition (E. Aichinger and N.M.)

Let $\mathbf{A} = (\mathbb{Z}_{11}, 2x + 3y)$, where $+$ is addition modulo 11 and $\cdot$ is multiplication modulo 11, and let $n \in \mathbb{N}$. Then there is a linear polynomial $p(n)$ such that $\gamma_{\mathbf{A}}(n) \leq p(n)$.

Theorem (G. Horváth and Ch. Nehaniv, 2014)

Let $n, k \in \mathbb{N}$ and $\mathbf{G}$ be a finite k-nilpotent group. Then $\gamma_{\mathbf{G}}(n) \leq c \cdot n^k$, where c is a constant that depends on $\mathbf{G}$.

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Certain expanded groups

Expanded groups

An algebra (A, F) is called an expanded group if there exist group operations in F.

Ω-groups

An expanded group $(A, +, -, 0, F)$ is called an Ω-group if for all $f \in F$ we have $f(0, \dots, 0) = 0$.

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„Easy" expanded groups

Easy Ω-groups

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite group $(V, +, -, 0)$ with one additional unary operation $f : V \to V$ such that $f(0) = 0$.

Exponent

In $\mathbf{V}$ the group $(V, +, -, 0)$ has a finite exponent $expV$ because V is a finite set. We will denote $E = expV - 1$.

Repetition of the unary operation

Since V is a finite set there are $F \in \mathbb{N}$ and $k \leq F$ such that $f^{F+1} = f^k$. We choose the smallest such F.

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A bit more conditions

Remark

As G. Horváth and Ch. Nehaniv consider nilpotent groups we are going to consider that our easy expanded group $\mathbf{V}$ is supernilpotent.

Easy 3-supernilpoitent expanded group

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$.

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Commutators and higher commutators

In Groups

If H, K are normal subgroups of a group **G** then $[H, K]$ is a normal subgroup generated by $\{[h,k] \mid h \in H,\ k \in K\}$, where $[h,k] := h^{-1}k^{-1}hk$ for all $h \in H$ and $k \in K$.

A. Bulatov, 2001

The term condition n-ary commutator $[\underbrace{\bullet, \dots, \bullet}_{n}]$ in a Mal'cev algebra is an n-ary operation on Con **A**.

Definition (3-supernilpotent)

Mal'cev algebras that satisfy $[1,1,1,1] = 0$ are called 3-supernilpotent.

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The bound

Theorem (E. Aichinger, M. Lazić and N. M., 2014)

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$ and let $n \in \mathbb{N}$. Then $\gamma_{\mathbf{V}}(n) \leq an^3 + bn^2 + cn + d$, where
$a = (F+1)^3(24EF^3 + 90EF^2 + 107EF + 4F^2 + 5F + 50E + 2)$,
$b = \frac{1}{2}(F+1)^2(9EF^2 + 27EF + 2F + 18E + 2)$,
$c = \frac{1}{2}(F+1)(EF + 2E + 2)$ and $d = -(F+1)$.

Further research

The next task

Find a criteria to decide whether the bound is polynomial?

Conjecture

In case of Mal'cev algebras the bound is polynomial if and only if the algebra is supernilpotent.

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Thank You for the Attention!