Quantum structures as near semirings

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- Łukasiewicz near semirings

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- Basic algebras as near semirings

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- Orthomodular lattices as near semirings

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Motivation: representing quantum structures as semirings, similarly to what is done for MV-algebras (Di Nola et al.)

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$\mathbf{R}$ is a semiring if $\cdot$ is also associative and it satisfies also Left distributivity, i.e. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

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Remark 1

If **R** is an idempotent near semiring. Then $\langle R, + \rangle$ is a (join) semilattice. If **R** is also integral then 1 is the top element with respect to the order $\leq$ induced by $+$.

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If **R** is an idempotent commutative semiring, whose multiplication is also idempotent ($x \cdot x = x$) and associative. Then clearly $\langle R, +, \cdot \rangle$ is a (non-distributive) bisemilattice

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Theorem 1

Let **R** be an involutive near semiring. Define two new operations as $x \mathbin{\widehat{+}} y = (x' + y')'$ and $x \mathbin{\widehat{\cdot}} y = (x' \cdot y')'$. Then $\mathbf{R}_\alpha = \langle R, \widehat{+}, \widehat{\cdot}, \ ', 0', 1' \rangle$ is a near semiring.

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(a) $x \cdot x' = x' \cdot x = 0$.

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(c) $x \leq y$ if and only if $x \cdot y' = 0$.

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Corollary 1

(a) Every Łukasiewicz semiring is commutative.

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Theorem 2

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(b) A Łukasiewicz near semiring is a Łukasiewicz semiring if and only if multiplication is associative.

Basic algebras

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Every basic algebra is a bounded lattice, where the order is defined as $x \leq y$ iff $x' \oplus y = 1$ and $x \vee y = (x' \oplus y) \oplus y$, $x \wedge y = (x' \vee y')'$. 0 and 1 are the bottom and the top elements of the lattice.

Basic algebras as near semirings

Theorem 3

Let R be a Łukasiewicz near semiring. Let $a \in R$, then the map $h_a : [a, 1] \to [a, 1]$, defined as $x \longmapsto x^a = (x \cdot a')'$ is an antitone involution in the interval $[a, 1]$.

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Let **R** be a Łukasiewicz near semiring. By defining
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Theorem 4

Let $\mathbf{R}$ be a Łukasiewicz near semiring. By defining $x \oplus y = ((x' + y) \cdot y')'$, then $\mathbf{B}(\mathbf{R}) = \langle R, \oplus, ', 0 \rangle$ is a basic algebra.

MV-algebras and near semirings

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Corollary

Let $\mathbf{M} = \langle M, \oplus, ', 0\rangle$ an MV-algebra. Then $\mathbf{R}(\mathbf{M})$ is a commutative Lukasiewicz near semiring.

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Corollary

Let $\mathbf{M} = \langle M, \oplus, ', 0 \rangle$ an MV-algebra. Then $\mathbf{R}(\mathbf{M})$ is a commutative Lukasiewicz near semiring.
Let $\mathbf{R} = \langle R, +, \cdot, ', 0, 1 \rangle$ be a Lukasiewicz semiring and let
$x \oplus y = ((x' + y) \cdot y')'$. Then $\mathbf{M}(\mathbf{R}) = \langle R, \oplus, \alpha, 0 \rangle$ is an MV-algebra.

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- $x \leq y \;\Rightarrow\; y = x \vee (y \wedge x')$. (Orthomodular law)

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The orthomodular law can be equivalently expressed by an identity, for example $(x \vee y) \wedge (x \vee (x \vee y)') = y$.

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Lemma

In an orthomodular near semiring, the following holds:

- $x \cdot x = x$
- $x + x' = 1$

OML as near semirings

Theorem 6

Let $\mathbf{L} = \langle L, \vee, \wedge, ', 0, 1 \rangle$ be an orthomodular lattice and define multiplication as $x \cdot y := (x \vee y') \wedge y$ (the Sasaki hook).

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Let $\mathbf{R}$ be an orthomodular near semiring. Setting $x \vee y = x + y$, and then defining $x \wedge y = (x' \vee y')'$,

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The variety of Lukasiewicz near semirings is congruence regular, witnessed by the terms

$$t_1(x, y, z) = (x \cdot y') + (y \cdot x') + z$$

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Theorem 9

The variety of Lukasiewicz near semirings is arithmetical.

The end!

Thanks for your attention!!