Robinson–Amitsur ultrafilters

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A theorem from 1960s

Theorem (S. Amitsur, A. Robinson)

If a prime associative ring R embeds in a direct product of associative division rings, then R embeds in an associative division ring.

Proof

Given embedding: $R \subseteq \prod_{i \in \mathbb{I}} A_i$.

$\mathcal{S} = \{\{i \in \mathbb{I} \mid f_i \neq 0\} \mid f \in R, f \neq 0\}$.

Primeness of $R \Rightarrow$ finite intersection property of $\mathcal{S} \Rightarrow \mathcal{S}$ extends to an ultrafilter $\mathcal{U}$.

$R \subseteq \prod_{\mathcal{U}} A_i \quad +$ Łoś' theorem.

A generalization from 2010s

Theorem ("Robinson–Amitsur for algebraic systems")

For any algebraic system A the following are equivalent:

(i) A is finitely subdirectly irreducible;

(ii) For any set $\{B_i\}_{i\in\mathbb{I}}$ of algebraic systems,
$A \subseteq \prod_{i\in\mathbb{I}} B_i \ \Rightarrow\ \exists$ ultrafilter $\mathscr{U}$ on $\mathbb{I}$: $A \subseteq \prod_{\mathscr{U}} B_i$.

Remark

For rings and algebras, primeness $\Rightarrow$ finite subdirect irreducibility.

Birkhoff meets Robinson–Amitsur

Criterion for absence of nontrivial identities

Let $\mathfrak{V}$ be a variety of algebraic systems such that any free system in $\mathfrak{V}$ is finitely subdirectly irreducible. Then for an algebraic system $A \in \mathfrak{V}$ the following are equivalent:

(i) A does not satisfy nontrivial identities within $\mathfrak{V}$;

(ii) any free system of $\mathfrak{V}$ embeds in an ultrapower of A;

(iii) any free system of $\mathfrak{V}$ embeds in a system elementarily equivalent to A.

Proof

(i) $\Rightarrow$ (ii) follows from Birkhoff's theorem + Robinson–Amitsur.
(ii) $\Rightarrow$ (iii) follows from Łoś' theorem.
(iii) $\Rightarrow$ (i) is trivial.

Applicable to:

All groups, Burnside varieties of groups, all algebras, associative algebras, Lie algebras.

Semigroups?

Question

What about semigroups? Inverse semigroups? Burnside varieties of semigroups? etc...

An obstacle

Free semigroups are not finitely subdirectly irreducible.

Applications

“Baby” Regev’s theorem

If A is a finite-dimensional associative algebra, and B is PI, then $A \otimes B$ is PI.

Algebras with the same identities (Kushkulei, Razmyslov, et al.)

If $\mathfrak{g}_1, \mathfrak{g}_2$ are finite-dimensional simple objects in some classes of algebras (Lie, Jordan, etc.), then $Var(\mathfrak{g}_1) = Var(\mathfrak{g}_2) \Leftrightarrow \mathfrak{g}_1 \simeq \mathfrak{g}_2$.

Growth sequence of Tarski’s monsters

Under some additional assumptions, the growth sequence (number of generators of $\underbrace{G \times \cdots \times G}_{n \text{ times}}$) of Tarski’s monster G is constant, equal to 2.

Another generalization

Theorem ("Robinson–Amitsur: from ω to κ")

For any algebraic system A, and any cardinal $\kappa > 2$ such that any κ-complete filter can be extended to a κ-complete ultrafilter, the following are equivalent:

(i) A is κ-subdirectly irreducible;

(ii) For any set $\{B_i\}_{i \in \mathbb{I}}$ of algebraic systems, $A \subseteq \prod_{i \in \mathbb{I}} B_i \Rightarrow \exists$ κ-complete ultrafilter $\mathscr{U}$ on $\mathbb{I}$: $A \subseteq \prod_{\mathscr{U}} B_i$.

A disappointment

No corollary similar to criterion for absence of nontrivial identities (second-order logic, big cardinals, ...)

Dual situation

Theorem (Bergman–Nahlus)

For any algebraic system A, and any cardinal $\kappa > 2$, the following are equivalent:

(i) For any surjective homomorphism $f : \prod_{i \in \mathbb{I}} B_i \to A$, $|\,\mathbb{I}\,| < \kappa$, there is $i_0 \in \mathbb{I}$ such that f factors through the canonical projection $\prod_{i \in \mathbb{I}} B_i \to B_{i_0}$.

(ii) For any surjective homomorphism $f : \prod_{i \in \mathbb{I}} B_i \to A$, there is a κ-complete ultrafilter $\mathscr{U}$ on $\mathbb{I}$ such that f factors through the canonical homomorphism $\prod_{i \in \mathbb{I}} B_i \to \prod_{\mathscr{U}} B_i$.

More questions

Question (Zilber)

Whether an ultraproduct of finite groups can be mapped surjectively on $SO(3)$?

Remark

By Bergman–Nahlus, "ultraproduct" can be replaced by "direct product".

Another question

Robinson–Amitsur for metric ultraproducts?

(Related to sofic groups, continuous first-order logic, etc.)

Based on:

- On the utility of Robinson-Amitsur ultrafilters, J. Algebra **388** (2013), 268–286; arXiv:0911.5414
- On the utility of Robinson-Amitsur ultrafilters. II, arXiv:1508.07496

Slides at `http://www1.osu.cz/~zusmanovich/math.html`

That's all. Thank you.