Equational properties of stratified least fixed points

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Fixed points in Computer Science

The semantics of recursion is usually described by fixed points of functions, functors, constructors, or morphisms.

Fixed points have been used: automata and language theory, semantics of programming languages, recursive data types, process algebra, programming logics, verification, computational complexity,...

Existence, construction and **logic** of fixed points.

Many people contributed: Bekić, De Bakker, Scott, Goguen, Thatcher, Wagner, Wright, Elgot, Eilenberg, Smyth, Plotkin, Niwinski, Courcelle, Nivat, Bloom, ...

A common basic theory of fixed point operations:

Iteration theories or **Iteration categories**
Bloom–Elgot–Wright (1980), ZE(1980)

Prototypical example: the category of complete lattices and monotonic functions, equipped with the parametrized least fixed point operation.

The parametrized least fixed point operation

A, B complete lattices, $f : A \times B \to A$ monotonic. Then for each fixed $y \in B$, the fixed point equation

$$x = f(x, y)$$

has a least solution $f^\dagger(y)$. The function $f^\dagger : B \to A$ is monotonic.

Parametric fixed point operator

$$f : A \times B \to A \quad \mapsto \quad f^\dagger : B \to A$$

Some identities

Fixed point identity

$$f^\dagger = f \circ \langle f^\dagger, \mathsf{id}_B \rangle : B \to A, \qquad f : A \times B \to A$$

$$x = f(x, y)$$
$$x = f^\dagger(y)$$

Parameter identity

$$(f \circ (\mathsf{id}_A \times g))^\dagger = f^\dagger \circ g : C \to A, \qquad f : A \times B \to A, g : C \to B$$

$$x = f(x, g(z)) = h(x, z) \qquad\qquad x = f(x, y)$$
$$x = h^\dagger(z) \qquad\qquad x = f^\dagger(y)$$
$$h^\dagger(z) = f^\dagger(g(z))$$

Some identities

Double dagger identity

$$f^{\dagger\dagger} = (f \circ (\langle \mathsf{id}_A, \mathsf{id}_A \rangle \times \mathsf{id}_B))^\dagger : B \to A, \quad f : A \times A \times B \to A$$

$$\begin{array}{ll} x = f(x, y, z) & x = f(x, x, z) = g(x, z) \\ x = f^\dagger(y, z) & x = g^\dagger(z) \\ y = f^\dagger(y, z) & \\ y = f^{\dagger\dagger}(z) & \end{array}$$

$$f^{\dagger\dagger}(z) = g^\dagger(z)$$

Completeness

Complete description: axioms of **iteration theories/categories** (ZE 1980)

- No finite equational base (Bloom–ZE 2000)
- Simplest known base: some classical identities + generalized power identities (ZE 1999, 2015)
- Finite quasi-equational axiomatizations
  - $f^{\dagger\dagger} = g^{\dagger\dagger} \Rightarrow f^{\dagger\dagger} = (f \circ \langle g^\dagger, \mathsf{id}_{A\times B}\rangle)^\dagger, \quad f, g : A \times A \times B \to A$ (Bloom–ZE 1993)
  - $f \circ \langle g, \mathsf{id}_B\rangle = g \Rightarrow f^\dagger \leq g, \quad f : A \times B \to A, g : B \to A$ (ZE 1996)

Fixed points of non-monotonic functions

Sometimes we need to solve fixed point equations involving **non-monotonic** functions.

Logic programming, language equations, boolean automata, synchronous and asynchronous circuits, etc.

Two major approaches

Bilattices: Denecker, Fitting, van Gelder, Trusczyński,...

Stratified complete lattices: Rondogiannis, Wadge, ...

Stratified complete lattices

$L = (L, \leq)$ a complete lattice, $(\sqsubseteq_\alpha)_{\alpha<\kappa}$ is a family of preoredrings of L.

Axiom 1 $\forall \beta < \alpha < \kappa$: $\sqsubseteq_\alpha$ is included in $=_\beta$.

Axiom 2 $\bigcap_{\alpha<\kappa} =_\alpha$ is the equality relation.

Axiom 3 $\forall x \in L$, $\alpha < \kappa$, $X \subseteq (x]_\alpha = \{z : \forall \beta < \alpha \ z =_\beta x\}$ $\exists y \in (x]_\alpha$:

1. $X \sqsubseteq_\alpha y$
2. $\forall z \in (x]_\alpha \ X \sqsubseteq_\alpha z \Rightarrow (y \sqsubseteq_\alpha z \ \wedge \ y \leq z)$.

The element y is unique: $y = \bigsqcup_\alpha X$.

Axiom 4 $\forall X \subseteq L, X \neq \emptyset, \forall y \in L, \alpha < \kappa : y =_\alpha X \Rightarrow y =_\alpha \bigvee X$.

Stratified complete lattices will be called **models**, for short.

The standard model

$$V: \quad F_0 < F_1 < \ldots < F_\alpha < \ldots < 0 < \ldots < T_\alpha < \ldots < T_1 < T_0, \quad \alpha < \Omega$$

For any set Z, $(V^Z, \leq)$, equipped with the pointwise order is a complete lattice: $f \leq g$ iff $\forall z \; f(z) \leq g(z)$.

For each ordinal $\alpha < \Omega$, define $f \sqsubseteq_\alpha g$ iff:

1. f and g agree up to (but not including) level α:

$$\forall z \forall \beta < \alpha \quad f(z) = F_\beta \Leftrightarrow g(z) = F_\beta \quad \wedge \quad f(z) = T_\beta \Leftrightarrow g(z) = T_\beta$$

2. f is below g at level α:

$$\forall z \quad f(z) = T_\alpha \Rightarrow g(z) = T_\alpha \quad \wedge \quad g(z) = F_\alpha \Rightarrow f(z) = F_\alpha$$

Lattice theorem

Define

$$x \sqsubseteq y \quad \text{iff} \quad x = y \ \vee \ \exists \alpha < \kappa \ x \sqsubset_\alpha y$$

Theorem (**Lattice theorem**, ZE-Rondogiannis) If L is a model, then $(L, \sqsubseteq)$ is a complete lattice.

Least element: $\perp$, greatest element not necessarily $\top$

Weakly monotonic functions

$f : L \to L'$, where L, L' models.

f α-**monotonic**, where $\alpha < \kappa$, if

$$\forall x, y \in L \quad x \sqsubseteq_\alpha y \;\Rightarrow\; f(x) \sqsubseteq_\alpha f(y)$$

f **weakly monotonic** if it is α-monotonic for all $\alpha < \kappa$.

Example Let $\kappa = \Omega$ and let $\neg : V^Z \to V^Z$:

$$(\neg f)(x) = \begin{cases} T_{\alpha+1} & \text{if } f(x) = F_\alpha \\ F_{\alpha+1} & \text{if } f(x) = T_\alpha \\ 0 & \text{if } f(x) = 0 \end{cases}$$

Let P be a normal logic program over Z. One can canonically associate a function $f_P : V^Z \to V^Z$ with P. Then f_P is weakly monotonic.

Fixed point theorem

Theorem (**Fixed point theorem**, ZE-Rondogiannis) L a model and $f : L \to L$ weakly monotonic. Then there is a $\sqsubseteq$-least $x \in L$ with $f(x) \sqsubseteq x$. Moreover, this least pre-fixed point x is a fixed point.

Remark

- f is not necessarily monotonic w.r.t. $\sqsubseteq$.
- When $\sqsubseteq_0 = \leq$ and $\sqsubseteq_\alpha$ is the equality relation on L, for all $0 < \alpha < \kappa$, then $\sqsubseteq$ is the ordering $\leq$, a function $f : L \to L$ is weakly monotonic iff it is monotonic w.r.t. $\leq$, and the theorem asserts that when $f : L \to L$ is monotonic, then it has a least (pre-)fixed point w.r.t. $\leq$.

Parameters

But we also want to solve equations $x = f(x, y)$ involving parameters.

Fact Models and weakly monotonic functions form a Cartesian category.

Proposition Suppose that L, L' are models and $f : L \times L' \to L$ is weakly monotonic. Then for each $y \in L'$, there is a $\sqsubseteq$-least $x = f^\dagger(y)$ with $f(x, y) \sqsubseteq x$. Moreover, it holds that $x = f(x, y)$ and $f^\dagger : L' \to L$ is weakly monotonic.

So we have a (parametrized) **stratified least fixed point operation**:

$$f : A \times B \to A \quad \mapsto \quad f^\dagger : B \to A$$

It extends the (parametrized) **least fixed point operation** on monotonic functions of complete lattices.

Main result

Theorem (ZE) An identity holds for the stratified least fixed point operator over weakly monotonic functions of models iff it holds in iteration categories.

Further results

- α-continuous functions and weakly continuous functions.

- Cartesian closed categories and the abstraction identity.

- A stronger notion of models. Fixed points form a complete lattice.

- Representation theorem by inverse limit models.