TECHNISCHE
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WIEN

Centralisers in algebra and elsewhere

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Disclaimer

Today all my carrier sets A are finite and non-empty!

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Assume $A = \{0, \ldots, k-1\}$ if it helps.

Apologies

I will bore you.

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Sit back, relax and

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Toy example $G = \mathrm{Dih}(3)$

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- $x \perp y \iff y \perp x$

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- these are all possible subgroups,
  so no other centralisers (not a general phenomenon)

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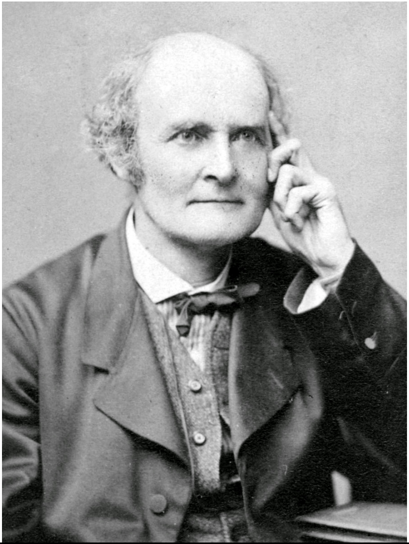
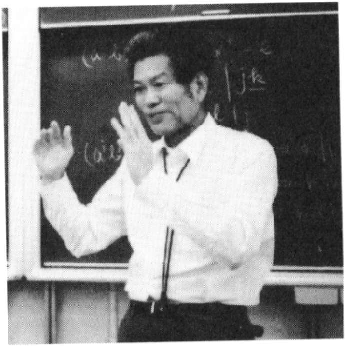
. . . but first quiz: who is this?





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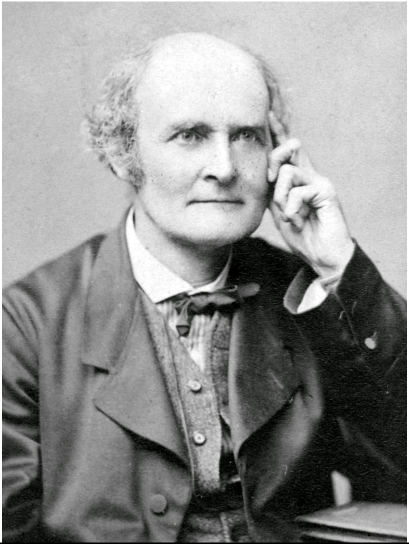

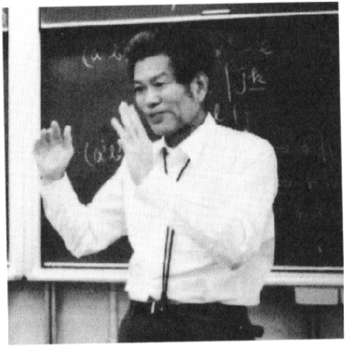

米田信夫

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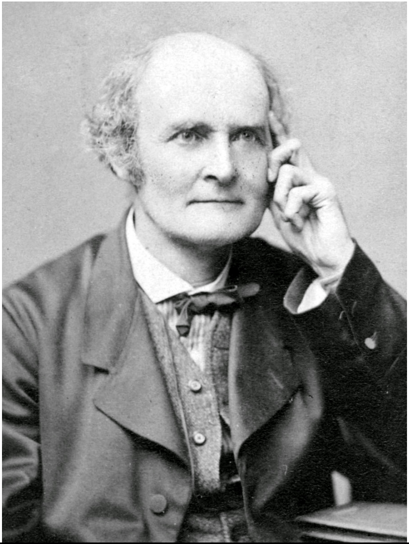
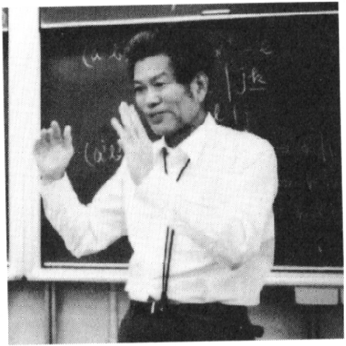
Arthur Cayley



Nobuo Yoneda



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Cayley (or Yoneda) says. . .

- every group is (isomorphic to) a permutation group
- every semigroup is a transformation semigroup
- every monoid is a transformation monoid
- every Menger algebra is a Menger algebra of functions
- every abstract clone is a concrete clone

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$M \leq \langle T(A); \circ \rangle$ transformation monoid

$$f^* = \{g \in M \mid g \circ f = f \circ g\} \qquad (f \in M)$$
$$F^* = \{g \in M \mid \forall f \in F\colon g \circ f = f \circ g\} \qquad (F \subseteq M)$$
$$f \perp g \iff \forall x \in A\colon g(f(x)) = f(g(x)) \qquad (f, g \in M)$$

I like higher-ary operations even more. . .

Finitary operations

- For $n \in \mathbb{N}_+$ a func $f\colon A^n \longrightarrow A$ is a n-ary operation on A
- $\mathcal{O}_A^{(n)} := A^{A^n}$ set of n-ary operations on A
- $\mathcal{O}_A := \bigcup_{n\in\mathbb{N}_+} \mathcal{O}_A^{(n)}$ set of all finitary operations on A

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No nullary operations, so sad. . .

Commutation for higher-ary operations

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For $n, m \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$, we define

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Commutation for higher-ary operations

Commutation

For $n, m \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$, we define

$$f \perp g \;:\Longleftrightarrow\; \forall\, X = (x_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \in A^{m\times n}:$$

$$\begin{array}{cccccc} & g & & g & & \\ & \frown & & \frown & & \\ f(& x_{1,1} & \cdots & x_{1,n}) & = & r_1 \\ & \vdots & \ddots & \vdots & & \vdots \\ f(& x_{m,1} & \cdots & x_{m,n}) & = & r_m \\ & \smile & & \smile & & \\ & = & & = & & \\ & c_1 & \cdots & c_n & & \end{array}$$

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 & \smile & & \smile & & \smile\\
 & = & & = & & =\\
f(& c_1 & \cdots & c_n &) = & y
\end{array}$$

Remarks on the definition

Remark 1

Defining for $m, n \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$ and any $X = (x_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \in A^{m\times n}$

$$f(X) := \begin{pmatrix} f\left(X\left(1,\cdot\right)\right) \\ \vdots \\ f\left(X\left(m,\cdot\right)\right) \end{pmatrix} = \begin{pmatrix} f\left(x_{1,1},\ldots,x_{1,n}\right) \\ \vdots \\ f\left(x_{m,1},\ldots,x_{m,n}\right) \end{pmatrix}$$

(f row-wise)

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(f row-wise)

we have for $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$:

$$f \perp g \iff \forall Y = (y_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in A^{m \times n} \colon g\left(f\left(Y\right)\right) = f\left(g\left(Y^T\right)\right).$$

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Corollary (since $\left(X^T\right)^T = X$)

For $f, g \in \mathcal{O}_A$: $f \perp g \iff g \perp f$.

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For $n, m \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$:
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For $n, m \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$:

$$f \perp g \iff \langle A; f, g\rangle \models \quad f\left(g\left(X\left(\cdot, 1\right)\right), \dots, g\left(X\left(\cdot, n\right)\right)\right) \approx g\left(f\left(X\left(1, \cdot\right)\right), \dots, f\left(X\left(1, \cdot\right)\right)\right)$$

Preservation of relations

Relations

Any $\varrho \subseteq A^m$ m-ary relation on A

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Functions preserve relations

For $n, m \in \mathbb{N}_+$, $f \in \mathcal{O}_A^{(n)}$, $\varrho \subseteq A^m$

$$f \triangleright \varrho :\Longleftrightarrow$$

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Any $\varrho \subseteq A^m$ — *m*-ary relation on *A*

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Relation to commutation

Graph of $g \in \mathcal{O}_A^{(m)}$

$$g^\bullet = \{(\boldsymbol{x}, g(\boldsymbol{x})) \mid \boldsymbol{x} \in A^m\} \subseteq A^{m+1}$$

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g & & g \\
\frown & & \frown \\
x_{1,1} & \cdots & x_{1,n} \\
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\smile & & \smile \\
= & & = \\
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 & \underbrace{} & & \underbrace{} & \\
 & = & & = & \\
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 & \Cap & & \Cap & \\
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 & g^\bullet & \cdots & g^\bullet & \Rightarrow & g^\bullet
\end{array}$$

$$\Longleftrightarrow\; f \perp g$$

Galois connections

Pol - Inv Galois correspondence

For $F \subseteq \mathcal{O}_A$ (operations) and $Q \subseteq \mathcal{R}_A$ (relations) we put

$$\mathrm{Pol}_A\, Q := \{ f \in \mathcal{O}_A \mid \forall \varrho \in Q\colon f \rhd \varrho \} \quad \text{(polymorphisms of } Q\text{)}$$

$$\mathrm{Inv}_A\, F := \{ \varrho \in \mathcal{R}_A \mid \forall f \in F\colon f \rhd \varrho \} \quad \text{(invariants of } F\text{)}$$

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$\mathrm{Pol}_A\,\mathrm{Inv}_A\, F$ clone gen. by F (closure: composition, projs)
$\mathrm{Inv}_A\,\mathrm{Pol}_A\, Q$ relational clone gen by Q (closure: pp-definitions)

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Centraliser Galois correspondence

For $F \subseteq \mathcal{O}_A$ we put

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F^{**} bicentraliser of F (closure: (clone +) pp-definitions)

Connections

Centralisers & bicentralisers

For $F \subseteq \mathcal{O}_A$ we have $\quad (F^\bullet := \{ f^\bullet \mid f \in F\})$

$F^* = \operatorname{Pol}_A F^\bullet \quad$ (centralisers are clones!)

Centralisers & bicentralisers

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$$
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&= \{g \in \mathcal{O}_A \mid g^\bullet \in \operatorname{Inv}_A F^*\} = \{g \in \mathcal{O}_A \mid g^\bullet \in \operatorname{Inv}_A \operatorname{Pol}_A F^\bullet\}
\end{aligned}
$$

Complete lattices

$\mathcal{L}_A$ — clones

$\mathcal{C}_A \subseteq \mathcal{L}_A$ — centraliser clones ($\bigwedge$-subsemilattice)

Knowledge about these lattices

$|A| = 2$

$|\mathcal{L}_A| = \aleph_0$
(E. L. Post, 1921/1941)

$|A| = 2$

$|\mathcal{C}_A| = 25$
(А. В. Кузнецов, 1977; et. al.)

$|A| \geq 3$

$|\mathcal{L}_A| = 2^{\aleph_0}$ (Ю. И. Янов / А. А. Мучник, 1959)

$|A| = 3$

$|\mathcal{C}_A| = 2986$
(А. Ф. Данильченко, 1974–79)

$|A| \geq 3$

minimal and maximal clones
(I. G. Rosenberg, 1970,1986)

$|A| < \aleph_0$

$|\mathcal{C}_A| < \aleph_0$
S. Burris, R. Willard, 1987

different bits and pieces

Unary generated

(W. Harnau, 1974–76;
I. G. Rosenberg, H. Machida)

We know the basics about centralisers in algebra. . .

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. . . now lets move on to “elsewhere”.

The algebraist's take on CSP

Γ — a finite relational signature

$\underset{\sim}{\mathbf{T}}$ — a finite relational structure of signature Γ (template)

CSP($\underset{\sim}{\mathbf{T}}$)... a decision problem

Instance: $\underset{\sim}{\mathbf{V}}$ of signature Γ

Question: $\mathrm{Hom}\left(\underset{\sim}{\mathbf{V}}, \underset{\sim}{\mathbf{T}}\right) \neq \emptyset$?

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Computer Scientists have a different perspective on CSP($\underset{\sim}{\mathbf{T}}$)

Instance: $\underset{\sim}{\mathbf{V}} = \left\langle V; \left(R^{\underset{\sim}{\mathbf{V}}}\right)_{R\in\Gamma} \right\rangle \leftrightarrow \varphi := \bigwedge_{R\in\Gamma} \bigwedge_{\boldsymbol{v}\in R^{\underset{\sim}{\mathbf{V}}}} R(\boldsymbol{v})$

Question: $\exists s\colon V \longrightarrow T\colon\ (\underset{\sim}{\mathbf{T}}, s) \models \varphi$?

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CSP($\underset{\sim}{\mathbf{T}}$)… a decision problem

Instance: $\underset{\sim}{\mathbf{V}}$ of signature Γ

Question: $\mathrm{Hom}\left(\underset{\sim}{\mathbf{V}}, \underset{\sim}{\mathbf{T}}\right) \neq \emptyset$?

Computer Scientists have a different perspective on CSP($\underset{\sim}{\mathbf{T}}$)

Instance: $\underset{\sim}{\mathbf{V}} = \left\langle V; \left(R^{\underset{\sim}{\mathbf{V}}}\right)_{R\in\Gamma} \right\rangle \leftrightarrow \varphi := \bigwedge_{R\in\Gamma} \bigwedge_{\boldsymbol{v}\in R^{\underset{\sim}{\mathbf{V}}}} R(\boldsymbol{v})$

Question: $\exists s\colon V \longrightarrow T\colon \left(\underset{\sim}{\mathbf{T}}, s\right) \models \varphi$?

Logicians still have another view on CSP($\underset{\sim}{\mathbf{T}}$)

Instance: $\underset{\sim}{\mathbf{V}} = \left\langle V; \left(R^{\underset{\sim}{\mathbf{V}}}\right)_{R\in\Gamma} \right\rangle \leftrightarrow \psi := (\exists v)_{v\in V} \bigwedge_{R\in\Gamma} \bigwedge_{\boldsymbol{v}\in R^{\underset{\sim}{\mathbf{V}}}} R(\boldsymbol{v})$

Question: $\underset{\sim}{\mathbf{T}} \models \psi$, i.e., $\psi \in$ pp-Th($\underset{\sim}{\mathbf{T}}$)?

Example

$$\underset{\sim}{\mathbf{T}} = \left\langle \{0,1\}\,;R^{\mathbf{T}}_{\sim}, S^{\mathbf{T}}_{\sim} \right\rangle;\; R^{\mathbf{T}}_{\sim} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \; S^{\mathbf{T}}_{\sim} = \leq_2$$

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One instance, three perspectives

I1 $\underset{\sim}{\mathbf{V}} = \left\langle \{0,1,2\}\,;\, R^{\underset{\sim}{\mathbf{V}}}, S^{\underset{\sim}{\mathbf{V}}} \right\rangle$

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Q1 $\exists h \in \operatorname{Hom}(\underset{\sim}{\mathbf{V}}, \underset{\sim}{\mathbf{T}})$?

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Q2 $\exists h\colon V \longrightarrow T\colon\ (\underset{\sim}{\mathbf{T}}, h) \models \varphi$?

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I3 $\psi = \exists x_0 \exists x_1 \exists x_2 : \varphi$

Q3 $\psi \in \text{pp-Th}(\underset{\sim}{\mathbf{T}})$?

Connection to clones

$\underset{\sim}{\mathbf{T}}_1 = \langle T; Q_1 \rangle$, $\underset{\sim}{\mathbf{T}}_2 = \langle T; Q_2 \rangle$ templates

pp-definable templates

$$Q_1 \subseteq \mathrm{Inv}_T \, \mathrm{Pol}_T \, Q_2 \implies \mathrm{CSP}\left(\underset{\sim}{\mathbf{T}}_1\right) \leq_{\mathrm{m}} \mathrm{CSP}\left(\underset{\sim}{\mathbf{T}}_2\right)$$
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Reason: substitute pp-definitions (Jeavons, 1998)

$$\exists \boldsymbol{x} \bigwedge_i R_i(\boldsymbol{x}_i) \iff \exists \boldsymbol{x} \bigwedge_i \exists \boldsymbol{y}_i \bigwedge_{j(i)} S_{j(i)}((\boldsymbol{x}_i, \boldsymbol{y}_i) \circ \alpha_{j(i)})$$
$$\iff \exists \boldsymbol{x}\boldsymbol{y} \bigwedge_\ell S_\ell((\boldsymbol{x}\boldsymbol{y}) \circ \alpha_\ell)$$

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CSP dichotomy conjecture

Fix a CSP template $\underset{\sim}{\mathbf{T}}$.

Time complexity

CSP $(\underset{\sim}{\mathbf{T}})$ is always in $\mathtt{NP}$, some CSP's are in $\mathtt{P}$.

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CSP dichotomy conjecture (Feder/Vardi, 1998)

Schaefer, 1978: $|T| = 2 \Rightarrow \mathrm{CSP}\left(\underset{\sim}{\mathbf{T}}\right)$ in $\mathtt{P}$ xor $\mathtt{NP}$-complete.
Conjecture: This extends to all finite domains.

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CSP dichotomy conjecture holds for...

$|T| = 2$ Schaefer, 1978

$|T| = 3$ Bulatov, 2006

$|T| = 4$ Marković et. al., 2011–12

$5 \leq |T| \leq 7$ Жук, 2016

Making connection with centraliser clones

$\mathbf{T}$ — a finite algebra of finite signature

HomAlg($\mathbf{T}$) — Feder/Madelaine/Stewart, 2004

Instance: $\mathbf{A}$ of the same signature as $\mathbf{T}$

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$$\operatorname{Hom}(\underset{\sim}{\mathbf{G}}, \underset{\sim}{\mathbf{H}}) \neq \emptyset \iff \operatorname{Hom}(\lambda_2(\underset{\sim}{\mathbf{G}}), \lambda_2(\underset{\sim}{\mathbf{H}})) \neq \emptyset$$

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Theorem for any rel template $\underset{\sim}{\mathbf{R}}$ — FMS

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Consequences

$\delta\iota\chi(\mathcal{C})$

= class $\mathcal{C}$ of decision problems enjoys a
$\mathtt{P}$ vs $\mathtt{NP}$-complete complexity dichotomy.

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HomAlg ($\mathbf{A}$) $\equiv_{\mathrm{m}}$ CSP ($\langle A; F^{\bullet}\rangle$)

Interpretation

Bad news

Establishing dichotomy for CSP with function graphs (even only of 2 unary ops) is **as bad as** as establishing dichotomy for the whole relational CSP.

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To get dichotomy for the whole relational CSP it suffices to have it for CSP with function graphs (even only of 2 unary ops).

Complexity of CSP $(\langle A; F^{\bullet}\rangle)$ determined by $\mathrm{Pol}_A\, F^{\bullet} = F^{*}$.

Thus, let's come back to centraliser clones. . .

Thus, let's come back to centraliser clones...

...I am not going to solve CSP-dichotomy, though.

What I am going to do

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I have a dream

What I am going to do

I want to prove

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I want to prove a

Theorem

$$|A| = k \geq 3 \implies \forall F \in \mathcal{C}_A \colon F = F^{(k)^{**}}$$

What I am going to do

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Theorem, perhaps

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Theorem, perhaps, some day

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(Burris-Willard conjecture)

What I am going to do

I want to prove a

Theorem, perhaps, some day, but not today...

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I want to prove a

Theorem today. . .

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STOP
FASTEN
YOUR
SEAT BELTS

A lemma on invariant relations

$Q \subseteq \mathcal{R}_A$, $\varrho \in \mathcal{R}_A$

For $m \geq |\varrho|$ TFAE

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2. $\varrho \in \operatorname{Inv}_A \operatorname{Pol}_A^{(m)} Q$

Proof (sketch)

- $\operatorname{Inv}_A \operatorname{Pol}_A^{(m)} Q = \operatorname{Inv}_A \operatorname{Pol}_A^{(\leq m)} Q$
- If $\varrho \notin \operatorname{Inv}_A \operatorname{Pol}_A Q$, i.e. $\operatorname{Pol}_A \{\varrho\} \not\supseteq \operatorname{Pol}_A Q$, then $\exists f \in \operatorname{Pol}_A Q\colon \quad f \not\triangleright \varrho$.
- If f is a counterexample of large arity ($> m$), identification of variables gives a counterexample $\tilde{f}$ of smaller arity

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Generally

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$$f \in \Sigma^{**}$$

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Corollaries (continued)

Corollary 2 for $\Sigma \subseteq \mathcal{O}_A$, $n \in \mathbb{N}_+$

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$$\forall \Sigma \subseteq \mathcal{O}_A\colon \quad \Sigma^{*} = \langle\Sigma\rangle^{(k)^{*}} \cap \operatorname{Pol}_A Q_{\langle\Sigma\rangle}$$

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For $F \in \mathcal{C}_A$, $\ell \geq k^k$, $k \geq 4$, let $G := F^{(\ell)^{**}}$

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Characterisation

For $n \geq k$ TFAE

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Corollary for $F \in \mathcal{C}_A$

$$F^* = F^{(k)^*} \cap \operatorname{Pol}_A Q_{\left\langle F^{\left(k^k\right)} \right\rangle}$$

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$\implies$ Requires understanding of Q_F.

Commutation of binary operations

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Does the following equation hold?

$$f(g(t \quad , y \quad), g(f \quad , a \quad)) \approx$$
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Does the following equation hold?

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