Transition operators assigned to physical systems

Ivan Chajda and Jan Paseka

Palacký
University
Olomouc

2016

Introduction

In 1900, D. Hilbert formulated his famous 23 problems. In the problem number 6, he asked: “Can physics be axiomatized?" It means that he asked if physics can be formalized and/or axiomatized for to reach a logically perfect system forming a basis of precise physical reasoning. This challenge was followed by G. Birkhoff and J. von Neuman in 1930s producing the so-called logic of quantum mechanics. We are going to addopt a method and examples of D. J. Foulis, however, we are not restricted to the logic of quantum mechanics. We are focused on a general situation with a physical system endowed with states which it can reach. Our goal is to assign to every such a system the so-called transition operators completely determining its transition relation. Conditions under which this assignment works perfectly will be formulated.

We start with a formalization of a given physical system. Every physical system is described by certain quantities and states through them it goes. From the logical point of view, we can formulate propositions saying what a quantity in a given state is. Through the paper we assume that these propositions can acquire only two values, namely either TRUE of FALSE. It is in accordance with reasoning both in classical physics and in quantum mechanics. Denote by S the set of states of a given physical system $\mathcal{P}$. It is given by the nature of $\mathcal{P}$ from what state $s \in S$ the system $\mathcal{P}$ can go to a state $t \in S$. Hence, there exists a binary relation R on S such that $(s, t) \in R$. This process is called a *transition* of $\mathcal{P}$.

Besides of the previous, the observer of $\mathcal{P}$ can formulate propositions revealing our knowledge about the system. The truth-values of these propositions depend on states. For example, the proposition p can be true if the system $\mathcal{P}$ is in the state s_1 but false if $\mathcal{P}$ is in the state s_2. Hence, for each state $s \in S$ we can evaluate the truth-value of p, it is denoted by $p(s)$. As mentioned above, $p(s) \in \{0, 1\}$ where 0 indicates the truth-value FALSE and 1 indicates TRUE. The set of all truth-values for all propositions will be called the *table*. Denote by B the set of propositions about the physical system $\mathcal{P}$ formulated by the observer. We can introduce the order $\leq$ on B as follows:

$$\text{for } p, q \in B, p \leq q \text{ if and only if } p(s) \leq q(s) \text{ for all } s \in S.$$

One can immediately check that the contradiction, i.e., the proposition with constant truth-value 0, is the least element and the tautology, i.e., the proposition with the constant truth-value 1 is the greatest element of the ordered set $(B; \leq)$; this fact will be expressed by the notation $(B; \leq, 0, 1)$ for the bounded ordered set of propositions about $\mathcal{P}$.

We summarize our description as follows:

- every physical system $\mathcal{P}$ will be identified with the couple (B, S), where B is the set of propositions about $\mathcal{P}$ and S is the set of states on $\mathcal{P}$;
- the set S is equipped with a binary relation R such that $\mathcal{P}$ can go from s_1 to s_2 provided $(s_1, s_2) \in R$;
- the set B is ordered by values of propositions as shown above.

Example 1

At first, let us describe a very simple physical system which is a pendulum. The pendulum can be considered to be a point mass suspended from a string or rod of negligible mass. The observer can e.g. distinguish three states as follows:

- s_1 means that the pendulum is in the right equilibrium point,
- s_2 means that the pendulum is moving,
- s_3 means that the pendulum is in the left equilibrium point.

The transition relation R on the set $S = \{s_1, s_2, s_3\}$ is visualized as follows.

Figpendul

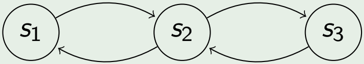

Figure 1: The transition graph of R

The set $B = \{0, p, q, r, p', q', r', 1\}$ of possible propositions on the physical system $\mathcal{P}$ is as follows:

- 0 means that the pendulum is in no state of S,
- p means that the pendulum is in the right equilibrium point,
- q means that the pendulum is moving,
- r means that the pendulum is in the left equilibrium point,
- 1 means that the pendulum is in at least one state of S.

Considering B as a classical logic (represented by a Boolean algebra), we can apply logical connectives conjunction $\wedge$, disjunction $\vee$, negation $'$ and implication $\Longrightarrow$ to create new propositions on $\mathcal{P}$. In our case, we can get e.g. $p' = q \vee r$ which means that the pendulum is either moving or in the left equilibrium point, etc. Altogether, we obtain eight propositions which can be ordered as follows in the Hasse diagram depicted in Figure 2.

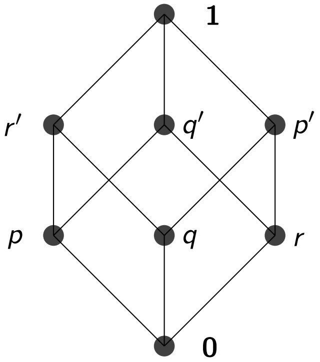

Figure 2: Logic of experimental propositions $\mathbb{B}$

Hence, our propositions form a Boolean algebra in accordance with our knowledge that $\mathcal{P}$ is a classical physical system and $\mathbb{B} = (B; \vee, \wedge, ', 0, 1)$ is a classical two-valued logic.
To shed another light on the previous concepts, let us addopt an example from [7].

Example 2

For system $\mathcal{P} = (B, S)$ we have
$B = \{0, l, r, n, f, b, l', r', n', f', b', 1\}$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$
such that the table is as follows.

	0	l	r	n	f	b
s_1	0	1	0	0	1	0
s_2	0	1	0	0	0	1
s_3	0	0	1	0	0	1
s_4	0	0	1	0	1	0
s_5	0	0	0	1	0	0

.

Remember that p' is a complement of p, i.e., it has 0 in the same instance where p has 1 and vice versa.

Then the order is vizualized in the following Hasse diagram depicted in Figure 3.

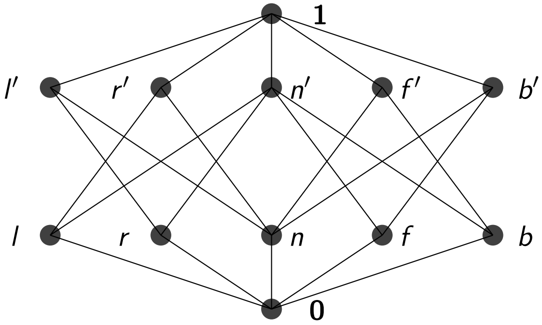

Figure 3: Logic of experimental propositions $\mathbb{L}$

Finally, the relation R on S is given as follows

$$R = \{(s_1, s_2), (s_2, s_3), (s_3, s_2), (s_3, s_5), (s_4, s_3), (s_4, s_5), (s_5, s_5)\}.$$

It can be vizualized by the following graph of transition.

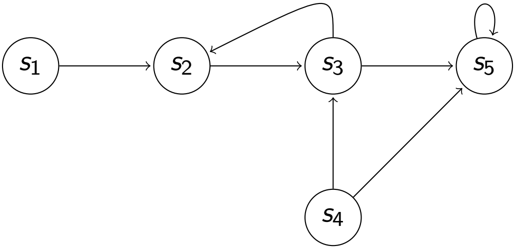

Figure 4: The transition graph of R

Our task is as follows. We introduce an operator T from B into 2^S which is constructed by means of the relation R. The question is if this operator, called *transition operator*, bears all the information about system $\mathcal{P}$ equipped with the relation R. In other words, if the relation R can be recovered by applying the operator T. In what follows, we will get conditions under which the transition operator has this property. Since these conditions are formulated in a pure algebraic way, we need to develop an algebraic background.

Algebraic Tools

Let S be a non-void set. Every subset $R \subseteq S \times S$ is called a *relation on* S and we say that the pair (S, R) is a *transition frame*. The fact that $(x, y) \in R$ for $x, y \in S$ is expressed by the notation xRy.

Let $(A; \leq)$ and $(B; \leq)$ be ordered sets, $f, g \colon A \to B$ mappings. We write $f \leq g$ if $f(a) \leq g(a)$, for all $a \in A$. A mapping f is called *order-preserving* or *monotone* if $a, b \in A$ and $a \leq b$ together imply $f(a) \leq f(b)$, and *order-reflecting* if $a, b \in A$ and $f(a) \leq f(b)$ together imply $a \leq b$. A bijective order-preserving and order-reflecting mapping $f \colon A \to B$ is called an *isomorphism*.

Let $(A; \leq)$ and $(B; \leq)$ be ordered sets. A mapping $f\colon A \to B$ is called *residuated* if there exists a mapping $g\colon B \to A$ such that

$$f(a) \leq b \quad \text{if and only if} \quad a \leq g(b)$$

for all $a \in A$ and $b \in B$.

In this situation, we say that f and g form a *residuated pair* or that the pair (f, g) is a (monotone) *Galois connection*.

Having a bounded poset $\mathbf{M} = (M; \leq, 0, 1)$ and a non-void set S, we can produce the direct power $\mathbf{M}^S = (A^S; \leq, 0, 1)$ where the relation $\leq$ is defined and evaluated on $p, q \in M^S$ componentwise, i.e. $p \leq q$ if $p(s) \leq q(s)$ for each $s \in S$. Moreover, $0, 1$ are such elements of M^S that $0(s) = 0$ and $1(s) = 1$ for all $s \in S$. Then $\mathbf{M}^S = (M^S; \leq, 0, 1)$ is a bounded poset as well. For any $s \in S$ and any $m = (m_t)_{t \in S} \in M^S$ we denote by $m(s)$ the s-th projection m_s.

We can take the following useful result from [3].

Observation 1 ([3])

Let $\mathbf{A}$ *and* $\mathbf{M}$ *be bounded posets,* S *a set and* $h_s \colon A \to M, s \in S$*, morphisms of bounded posets. The following conditions are equivalent:*

(i) $((\forall s \in S)\, h_s(a) \leq h_s(b)) \implies a \leq b$ *for any elements* $a, b \in A$*;*

(ii) *The map* $h \colon A \to M^S$ *defined by* $h(a) = (h_s(a))_{s \in T}$ *for all* $a \in A$ *is order reflecting.*

We then say that $\{h_s \colon A \to M; s \in S\}$ is a *full set of order-preserving maps with respect to* M. Note that we may in this case identify $\mathbf{A}$ with a bounded subposet of $\mathbf{M}^S$ since h is an order reflecting morphism alias *embedding* of bounded posets. Note that $h(a)(s) = h_s(a)$ for all $a \in A$ and all $s \in S$.

Consider a complete lattice $\mathbf{M} = (M; \leq, 0, 1)$ and let (S, R) be a transition frame. Further, let $\mathbf{A} = (A; \leq, 0, 1)$ be a bounded subposet of $\mathbf{M}^S$.
Define mappings $P_R : A \to M^S$ and $T_R : A \to M^S$ as follows: For all $A \in B$ and all $s \in S$,

$$T_R(b)(s) = \bigwedge_M \{b(t) \mid sRt\} \tag{$*$}$$

and, for all $a \in A$ and all $t \in S$,

$$P_R(a)(t) = \bigvee_M \{a(s) \mid sRt\}. \tag{$**$}$$

Then we say that T_R (P_R) is an *upper transition operator* (*lower transition operator*) *constructed by means of the transition frame* (S, R), respectively.

For to answer our question given in introduction, we introduce certain binary relations induced by means of operators mentioned above. Hence, let $\mathbf{A} = (A; \leq, 0, 1)$ be bounded poset with a full set S of morphisms of bounded posets into a non-trivial complete lattice $\mathbf{M}^S$. We may assume that $\mathbf{A}$ is a O bounded subposet of $\mathbf{M}^S$.

Let $P : A \to O$ and $T : A \to O$ be morphisms of posets. Let us define the relations

$$R_T = \{(s,t) \in S \times S \mid (\forall b \in B)(T(b)(s) \leq b(t))\} \qquad (\dagger)$$

and

$$R^P = \{(s,t) \in S \times S \mid (\forall a \in A)(a(s) \leq P(a)(t))\}. \qquad (\dagger\dagger)$$

The relations R_T and R^P on S will be called the *T-induced relation by* $\mathbf{M}$ and *P-induced relation by* $\mathbf{M}$, respectively.

Now, let let (S, R) be a transition frame and T_R, P_R operators constructed by means of the transition frame (S, R). We can ask under what conditions the relation R coincides with the relation R_{T_R} constructed as in (†) or with the relation R^{P_R} constructed as in (††). If this is the case we say that R *is recoverable from* T_R or that R *is recoverable from* P_R. We say that R is *recoverable* if it is recoverable both from T_R and P_R. The answer will be given in the next section.

The connection between the relations R and R_{T_R} or R and R^{P_R} is presented in the following lemma.

Lemma 1

Let $\mathbf{M}$ *be a non-trivial complete lattice and* S *a non-empty set. Let* $\mathbf{A}$ *be bounded subposet of* $\mathbf{M}^S$*. Then*

(a) $R_1 \subseteq R_2 \subseteq S \times S$ *implies* $T_{R_2} \leq T_{R_1}$ *and* $P_{R_1} \leq P_{R_2}$.

(b) *If* $T_1, T_2 : A \to M^S$ *are order-preserving mappings such that* $T_2 \leq T_1$ *then* $R_{T_1} \subseteq R_{T_2}$.

(c) *If* $P_1, P_2 : A \to M^S$ *are order-preserving mappings such that* $P_1 \leq P_2$ *then* $R^{P_1} \subseteq R^{P_2}$.

(d) If $R \subseteq S \times S$ then $R \subseteq R_{T_R} \cap R^{P_R}$.

(e) If $T : A \to M^S$ is an order-preserving mapping then $T \leq T_{R_T}$.

(f) If $P : A \to M^S$ is an order-preserving mapping then $P_{R^P} \leq P$.

(g) Both the extremal relations $S \times S$ and $\emptyset$ are recoverable from $T_{S\times S}$ and $T_\emptyset$ or $P_{S\times S}$ and $P_\emptyset$, respectively.

The connection between relations induced by means of transition operators T and P is shown in the following lemma and theorem. We obtain the relationship between transition operators and transition relations on a physical system $\mathcal{P} = (B, S)$.

Lemma 2

Let $\mathbf{M}$ *be a non-trivial complete lattice and* S *a non-empty set such that* $\mathbf{A}$ *is bounded subposet of* $\mathbf{M}^S$. *Let* $P : A \to M^S$ *and* $T : A \to M^S$ *be morphisms of posets such that, for all* $a \in A$ *and all* $b \in A$,

$$P(a) \leq b \iff a \leq T(b).$$

(a) *If* $P(A) \subseteq A$ *then* $R_T \subseteq R^P$.

(b) *If* $T(A) \subseteq A$ *then* $R^P \subseteq R_T$.

(c) *If* $P(A) \subseteq A$ *and* $T(A) \subseteq A$ *then* $R_T = R^P$.

In what follows, we are going to show that the transition relations on S and the transition operators on $\mathbf{B}$ form a Galois connection. This is important because in every Galois connection one of its components fully determines the second one and vice versa.

Among other things, the following theorem shows that if a given transition relation R can be recovered by the upper transition operator then, under natural conditions, it can be recovered by the lower transition operator and vice versa.

Theorem 1

Let $\mathbf{M}$ *be a non-trivial complete lattice and* (S, R) *a transition frame. Let* $\mathbf{A}$ *be a bounded subposet of* $\mathbf{M}^S$*. Let* $P_R : A \to M^S$ *and* $T_R : A \to M^S$ *be operators constructed by means of the transition frame* (S, R)*, i.e., the respective parts of the condition* $(\star)$ *hold for the relation* R*. Then, for all* $a \in A$ *and all* $b \in A$*,*

$$P_R(a) \leq b \iff a \leq T_R(b).$$

Moreover, the following holds.

(a) *Let for all* $t \in S$ *exist an element* $b^t \in A$ *such that, for all* $s \in S$*,* $(s, t) \notin R$*, we have* $\bigwedge_M \{b^t(u) \mid sRu\} \not\leq b^t(t) \neq 1$*. Then* $R = R_{T_R}$*.*

(b) *Let for all* $s \in S$ *exist an element* $a^s \in A$ *such that, for all* $t \in S$*,* $(s, t) \notin R$*, we have* $\bigvee_M \{a^s(u) \mid uRt\} \not\geq a^s(s) \neq 0$*. Then* $R = R^{P_R}$*.*

(c) *If* $R = R_{T_R}$ *and* $T_R(A) \subseteq A$ *then* $R = R_{T_R} = R^{P_R}$*.*

(d) *If* $R = R^{P_R}$ *and* $P_R(A) \subseteq A$ *then* $R = R_{T_R} = R^{P_R}$*.*

The following two corollaries of Theorem 1 show that if the set B of propositions on the system (B, S) is large enough, i.e., if it contains the full set $\{0,1\}^S$ then the transition relation R can be recovered by each of the transition operators.

Corollary 1

Let $\mathbf{M}$ be a non-trivial complete lattice and (S, R) a transition frame. Let $\mathbf{B}$ be a bounded subposets of $\mathbf{M}^S$ such that $\{0,1\}^S \subseteq B$. Let $P_R : B \to M^S$ and $T_R : B \to M^S$ be operators constructed by means of the transition frame (S, R), i.e., the respective parts of the condition $(\star)$ hold for the relation R. Then $R = R^{P_R} = R_{T_R}$.

Corollary 2

Let $\mathbf{M}$ be a non-trivial complete lattice and (S, R) a transition frame. Let $\widehat{T}, \widehat{P} : M^S \to M^S$ be operators constructed by means of (S, R). Then $R = R^{\widehat{P}} = R_{\widehat{T}}$.

We can illustrate previous results in the following two examples.

Example 3

Consider the system $\mathcal{P} = (B, S)$ of Example 2.
From the table we see the values of B as follows:

$$\begin{array}{ll}
0 = (0,0,0,0,0), & l = (1,1,0,0,0),\\
r = (0,0,1,1,0), & n = (0,0,0,0,1),\\
f = (1,0,0,1,0), & b = (0,1,1,0,0),\\
l' = (0,0,1,1,1), & r' = (1,1,0,0,1),\\
n' = (1,1,1,1,0), & f' = (0,1,1,0,1),\\
b' = (1,0,0,1,1), & 1 = (1,1,1,1,1).
\end{array}$$

Using our formulas $(*)$ and $(**)$, we compute the upper transition operator $T_R\colon B \to 2^S$ and the lower transition operator $P_R\colon B \to 2^S$ as follows:

Example 3

$$
\begin{array}{ll}
T_R(0) = 0, & T_R(1) = 1, \\
T_R(l) = (1,0,0,0,0), & T_R(l') = (0,0,0,1,1), \\
T_R(r) = (0,1,0,0,0), & T_R(r') = (1,0,1,0,1), \\
T_R(n) = n, & T_R(n') = l, \\
T_R(f) = 0, & T_R(f') = 1, \\
T_R(b) = l, & T_R(b') = n,
\end{array}
$$

$$
\begin{array}{ll}
P_R(0) = 0, & P_R(1) = f', \\
P_R(l) = b, & P_R(l') = f', \\
P_R(r) = f', & P_R(r') = f', \\
P_R(n) = n, & P_R(n') = f', \\
P_R(f) = f', & P_R(f') = f', \\
P_R(b) = f', & P_R(b') = f'.
\end{array}
$$

Hence, $T_R(l)$, $T_R(r)$, $T_R(l')$ and $T_R(r')$ do not belong to B, they only belong to 2^S. On the contrary, $P_R(B) \subseteq B$.

Now, we use T_R for computing the binary relation R_{T_R} (by the formula (†)) and P_R for computing the binary relation R^{P_R} (by the formula (††)). We obtain that $R_{T_R} = R$, i.e., our given relation R is recoverable by the transition operator T_R and hence this operator is a characteristic of the system $\mathcal{P} = (B, S)$.
On the contrary, R^{P_R} is given as follows

$$\begin{aligned} R^{P_R} \ &= \{(s_1, s_2), (s_1, s_3), (s_2, s_2), (s_2, s_3), (s_3, s_2), (s_3, s_3), (s_3, s_5), \\ &\quad (s_4, s_2), (s_4, s_3), (s_4, s_5), (s_5, s_5)\}. \end{aligned}$$

It can be vizualized by the following transition graph.

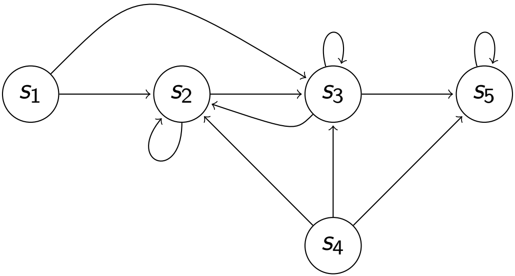

Figure 5: The transition graph of R^{P_R}

One can easily see that $R \subseteq R^{P_R}$ but $R = R_{T_R} \neq R^{P_R}$ in accordance with our Theorem 1 because the condition (b) from this theorem is not satisfied.
We can now use the same system $\mathcal{P} = (B, S)$ as above and, instead of the relation R on S, we apply the new relation $R_2 = R^{P_R}$. In other words, our system $\mathcal{P} = (B, S)$ will have the transition frame (S, R^{P_R}). After the computation of the new lower transition operator $P_{R_2} \colon B \to 2^S$ and the relation $R^{P_{R_2}}$ we see that now $R_2 = R^{P_{R_2}}$.
Let us compute the new upper transition operator $T_{R_2} \colon B \to 2^S$ and the relation $R_{T_{R_2}}$. It follows that

$$
\begin{array}{ll}
T_{R_2}(0) = 0, & T_{R_2}(1) = 1, \\
T_{R_2}(l) = 0, & T_{R_2}(l') = n, \\
T_{R_2}(r) = 0, & T_{R_2}(r') = n, \\
T_{R_2}(n) = n, & T_{R_2}(n') = l, \\
T_{R_2}(f) = 0, & T_{R_2}(f') = 1, \\
T_{R_2}(b) = l, & T_{R_2}(b') = n.
\end{array}
$$

Since $T_{R_2}(B) \subseteq B$ and $P_{R_2}(B) \subseteq B$ we obtain from Theorem 1 and Lemma 2 (c) that $R_{T_{R_2}} = R^{P_{R_2}} = R_2$, i.e., R_2 is recoverable both by the upper transition operator T_{R_2} and by the lower transition operator P^{R_2}.

The distinction of the classical physical system from the above disussed quantum one is that our logic $\mathbb{B}$ can easily satisfy the conditions $(\star)$ when taking $B = \{0,1\}^S$. This is valid in the case of our introductory example with a pendulum. When computing the transition operators T_R and P_R, they are restricted to B only (i.e., none of them goes out from B as in Example 3). In fact, we have

$$
\begin{array}{ll}
T_R(0) = 0, & P_R(0) = 0, \\
T_R(p) = 0, & P_R(p) = p', \\
T_R(q) = 0, & P_R(q) = q', \\
T_R(r) = 0, & P_R(r) = r', \\
T_R(p') = p, & P_R(p') = 1, \\
T_R(q') = q, & P_R(q') = 1, \\
T_R(r') = r, & P_R(r') = 1, \\
T_R(1) = 1, & P_R(1) = 1,
\end{array}
$$

i.e., $P_R =' \circ T_R \circ'$. Now, we can construct the transition relations R_{T_R} and R^{P_R} directly by (†) and (††). It is an easy exercise to check that really

$$R = R_{T_R} = R^{P_R}.$$

Hence, for a classical system, R can be recovered from the transition operators of the assigned logic $\mathbb{B}$ whenever $\mathbb{B}$ is large enough, i.e., if e.g. $B = \{0, 1\}^S$.

- BLYTH, T.S.: *Lattices and ordered algebraic structures*, Springer-Verlag London Limited, 2005.
- BURGES, J.: *Basic tense logic*. In: Handbook of Philosophical Logic, vol. II (D. M. Gabbay, F. Günther, eds.), D. Reidel Publ. Comp., 1984, pp. 89–139.
- CHAJDA, I.—PASEKA, J.: *Dynamic Effect Algebras and their Representations*, Soft Computing **16**, (2012), 1733–1741.
- CHAJDA, I.— PASEKA, J.: *Tense Operators and Dynamic De Morgan Algebras*, In: Proc. 2013 IEEE 43rd Internat. Symp. Multiple-Valued Logic, Springer, (2013), 219–224.
- CHAJDA, I.— PASEKA, J.: *Dynamic Order Algebras as an Axiomatization of Modal and Tense Logics*, International Journal of Theoretical Physics, doi: 10.1007/s10773-015-2510-9.
- FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24**, (1994), 1325–1346.

FOULIS, D. J.: *A half-century of quantum logic what have we learned?*, In Proc. Conf. Einstein Meets Magritte, Brussels, June, 1995.

HILBERT, D. : *Mathematical Problems*, Bull. Amer. Math. Soc. **8** (1902), 437–479.

The end!

Thank you for your attention!