ON TWO ORDERINGS OF MAXIMAL ORTHOGONAL SUBSETS OF AN ORTHOPOSET

Jānis Cīrulis

Institute of Mathematics and Computer Science

University of Latvia

email: jc@lanet.lv

The 54th Summer School on General Algebra and Ordered Sets

Trojanovice, September 3-9, 2016

ABOUT WHAT?

ABOUT WHAT?

Let P be an orthoposet.

Define two order relations on the set $\mathcal{M}$ of its maximal orthogonal subsets:

for $U, V \in \mathcal{M}$,

$U \leq_1 V$:≡ to every $v \in V$ there is $u \in U$ such that $v \leq u$

(U *supports* V),

$U \leq_2 V$:≡ every element of U is the join of a subset of V

(U is *coarser* than V) (V is a *refinement* of U).

ABOUT WHAT?

Let P be an orthoposet.

Define two order relations on the set $\mathcal{M}$ of its maximal orthogonal subsets:

for $U, V \in \mathcal{M}$,

$U \leq_1 V$:≡ to every $v \in V$ there is $u \in U$ such that $v \leq u$

(U *supports* V),

$U \leq_2 V$:≡ every element of U is the join of a subset of V

(U is *coarser* than V) (V is a *refinement* of U).

For all U, V, if $U \leq_2 V$, then $U \leq_1 V$. ⊣

Motivation

Motivation

(1) If P is orthocomplete, then with every $V \in M$ associated is a quantifier C_V, i.e., a closure operator on P whose range is a subalgebra of P:

$$\mathsf{C}_V(p) := \bigvee(v \in V : v \not\perp p).$$

Fact: $\mathsf{C}_U\mathsf{C}_V = \mathsf{C}_V$ iff $\mathsf{C}_V\mathsf{C}_U = C_V$ iff $U \leq_2 V$. ⊣

(2) In any set of variables X, a variable x *(functionally) depends* on y if

there is a function d: $\text{ran}\, y \to \text{ran}\, x$ such that

"every time" when y takes the value b, x takes the value $d(b)$.

(2) In any set of variables X, a variable x *(functionally) depends* on y if

there is a function d: ran $y \to$ ran x such that
"every time" when y takes the value b, x takes the value $d(b)$.

Call a *variable on* P any injective function defined on some $U \in \mathcal{M}$.

(2) In any set of variables X, a variable x *(functionally) depends* on y if

there is a function $d\colon \operatorname{ran} y \to \operatorname{ran} x$ such that
"every time" when y takes the value b, x takes the value $d(b)$.

Call a *variable on* P any injective function defined on some $U \in \mathcal{M}$. Then

a variable x depends on a variable y iff
to every $v \in \operatorname{dom} y$ there is $u \in \operatorname{dom} x$ such that $v \le u$ iff
$\operatorname{dom} x \le_1 \operatorname{dom} y$.

(2) In any set of variables X, a variable x *(functionally) depends* on y if

there is a function $d\colon \operatorname{ran} y \to \operatorname{ran} x$ such that
"every time" when y takes the value b, x takes the value $d(b)$.

Call a *variable on* P any injective function defined on some $U \in \mathcal{M}$. Then

a variable x depends on a variable y iff
to every $v \in \operatorname{dom} y$ there is $u \in \operatorname{dom} x$ such that $v \leq u$ iff
$\operatorname{dom} x \leq_1 \operatorname{dom} y$.

If x depends on y and x takes a value a, then y has a value in $d^{-1}(a)$, so the following should hold for every $u \in \operatorname{dom} x$:

$u \leq \bigvee(v \in \operatorname{dom} y\colon v \leq u)$.

All these joins exist iff $\operatorname{dom} x \leq_2 \operatorname{dom} y$. ⊣

Questions

(1) When do $\leq_1$ and $\leq_2$ coincide?
(2) When is $\mathcal{M}$ a bounded complete poset under $\leq_1$? under $\leq_2$?

1. ORTHOPOSETS

1. ORTHOPOSETS

An *orthoposet* is a quintuple $(P, \leq, ^{\perp}, 0, 1)$, where

- $(P, \leq, 0, 1)$ is a bounded poset,
- $^{\perp}$ is a unary operation (*orthocomplementation*),
- $p^{\perp\perp} = p, \quad p \leq q \Rightarrow q^{\perp} \leq p^{\perp}$,
- $0 = p \wedge p^{\perp}, \quad 1 = p \vee p^{\perp}$.

1. ORTHOPOSETS

An *orthoposet* is a quintuple $(P, \leq, ^{\perp}, 0, 1)$, where

- $(P, \leq, 0, 1)$ is a bounded poset,
- $^{\perp}$ is a unary operation (*orthocomplementation*),
- $p^{\perp\perp} = p, \quad p \leq q \Rightarrow q^{\perp} \leq p^{\perp}$,
- $0 = p \wedge p^{\perp}, \quad 1 = p \vee p^{\perp}$.

In an orthoposet P,

- elements p and q are said to be *orthogonal* (in symbols, $p \perp q$) if $p \leq q^{\perp}$ (equivalently, if $q \leq p^{\perp}$),
- a subset X is said to be *orthogonal* if elements of X differ from 0 and are mutually orthogonal. ⊣

Notation: for any $X \subseteq P$,

$X \leq q :\equiv x \leq q$ for all $x \in X$,

$X \perp q \equiv x \perp q$ for all $x \in X$.

Notation: for any $X \subseteq P$,

$X \leq q :\equiv x \leq q$ for all $x \in X$,

$X \perp q \equiv x \perp q$ for all $x \in X$.

An orthoposet is

- *orthocomplete* if every its orthogonal subset has a join,
- *Boolean* if $p \perp q$ whenever $p \wedge q = 0$.

Fact: If a Boolean orthoposet is a lattice or is orthocomplete, then it is a Boolean algebra.

2: COMPARING THE TWO ORDERS

2: COMPARING THE TWO ORDERS

In what follows, P is an arbitrary orthoposet, and $\mathcal{M}$ is the set of maximal orthogonal subsets of P. The letters U, V, W are reserved for elements of $\mathcal{M}$.

2: COMPARING THE TWO ORDERS

In what follows, P is an arbitrary orthoposet, and $\mathcal{M}$ is the set of maximal orthogonal subsets of P. The letters U, V, W are reserved for elements of $\mathcal{M}$.

Recall: for $U, V \in \mathcal{M}$,

$U \leq_1 V$:≡ to every $v \in V$ there is $u \in U$ such that $v \leq u$.

$U \leq_2 V$:≡ every element of U is the join of a subset of V.

2: COMPARING THE TWO ORDERS

In what follows, P is an arbitrary orthoposet, and $\mathcal{M}$ is the set of maximal orthogonal subsets of P. The letters U, V, W are reserved for elements of $\mathcal{M}$.

Recall: for $U, V \in \mathcal{M}$,

$U \leq_1 V$:≡ to every $v \in V$ there is $u \in U$ such that $v \leq u$.

$U \leq_2 V$:=≡ every element of U is the join of a subset of V.

Observations:

For all U, V if $U \leq_2 V$, then $U \leq_1 V$.

If P is orthocomplete, then $U \leq_2 V$ iff $U \leq_1 V$.

If P is Boolean, then $U \leq_2 V$ iff $U \leq_1 V$. ⊣

Elementary properties of $\leq_1$ and $\leq_2$.

Elementary properties of $\leq_1$ and $\leq_2$.

For given V, an element $p \in P$ is said to be

- *V-strict* if, for every $v \in V$, either $v \leq p$ or $v \perp p$,
- *V-exact* if p is a join of a subset of V.

Every V-exact element p is V-strict.

Elementary properties of $\leq_1$ and $\leq_2$.

For given V, an element $p \in P$ is said to be

- *V-strict* if, for every $v \in V$, either $v \leq p$ or $v \perp p$,
- *V-exact* if p is a join of a subset of V.

Every V-exact element p is V-strict.

$S(V)$:= the set of all V-strict elements,
$E(V)$:= the set of all V-exact elements.

Proposition

The following are equivalent in $\mathcal{M}$:

(a) $U \leq_1 V$,

(b) $U \subseteq S(V)$,

(c) $S(U) \subseteq S(V)$.

Proposition

The following are equivalent in $\mathcal{M}$:

(a) $U \leq_1 V$,

(b) $U \subseteq S(V)$,

(c) $S(U) \subseteq S(V)$.

Proposition

The following are equivalent in $\mathcal{M}$:

(a) $U \leq_2 V$,

(b) $U \subseteq E(V)$,

(c) $E(U) \subseteq E(V)$.

Proposition

The following are equivalent in $\mathcal{M}$:

(a) $U \leq_1 V$,
(b) $U \subseteq S(V)$,
(c) $S(U) \subseteq S(V)$.

Proposition

The following are equivalent in $\mathcal{M}$:

(a) $U \leq_2 V$,
(b) $U \subseteq E(V)$,
(c) $E(U) \subseteq E(V)$.

For any V, $E(V) \subseteq S(V)$.

When $S(V) = E(V)$? P orthocomplete or Boolean

An answer to Q1

An answer to Q1

A subset $V \in \mathcal{M}$ is *dense* if, for every p,

if $\{v \in V : v \not\perp p\} \leq q$, then $p \leq q$ for all q;

equivalently,

if $\{v \in V : v \not\perp p\} \perp q$, then $p \perp q$ for all q.

An answer to Q1

A subset $V \in \mathcal{M}$ is *dense* if, for every p,

if $\{v \in V : v \not\leq p\} \leq q$, then $p \leq q$ for all q;

equivalently,

if $\{v \in V : v \not\leq p\} \perp q$, then $p \perp q$ for all q.

Lemma

$S(V) \subseteq E(V)$ if and only if V is dense.

An answer to Q1

A subset $V \in \mathcal{M}$ is *dense* if, for every p,

if $\{v \in V : v \not\perp p\} \leq q$, then $p \leq q$ for all q;

equivalently,

if $\{v \in V : v \not\perp p\} \perp q$, then $p \perp q$ for all q.

Lemma

$S(V) \subseteq E(V)$ if and only if V is dense.

Theorem

The orders $\leq_1$ and $\leq_2$ coincide if and only if every maximal orthosubset of P is dense.

3. BOUNDED COMPLETENESS OF $\mathcal{M}$

3. BOUNDED COMPLETENESS OF $\mathcal{M}$

Preliminaries

$(X, \leq)$ — a poset.

X is *bouned complete* if
  every subset of X bounded above has the l.u.b.
This is the case iff
  every nonempty subset of X has the g.l.b.

3. BOUNDED COMPLETENESS OF $\mathcal{M}$

Preliminaries

$(X, \leq)$ — a poset.

X is *bouned complete* if
every subset of X bounded above has the l.u.b.
This is the case iff
every nonempty subset of X has the g.l.b.

In particular, then X is a *nearlattice* with bottom, i.e.,
- has the least element 0,
- any two elements of X have the meet, and
- every initial segment $[0, a]$ of X is a sublattice of X.

X is *locally bounded complete* if every subset of X having an upper bound a has the l.u.b. in the lower part $(a]$ of X.

Lemma

If X is locally bounded complete, then X is bounded complete.

For every bounded subset of X, its local join is its join in P.

An answer to Q2 . . .

An answer to Q2 . . .

Theorem

The poset $(\mathcal{M}, \leq_1)$ is bounded complete if and only if P is orthocomplete.

An answer to Q2 . . .

Theorem

The poset $(\mathcal{M}, \leq_1)$ is bounded complete if and only if P is orthocomplete.

. . . and some lemmas for it

(characterizing upper bounds of a subset of $\mathcal{M}$). ⊣

Recall: $U \leq_1 V :\equiv$ to every $v \in V$ there is $u \in U$ such that $v \leq u$.

Then $U \leq_1 V$ iff there is a mapping $\delta_U^V \colon V \to U$

such that $v \leq \delta_U^V(v)$.

If such a mapping exists, then it is unique.

Recall: $U \leq_1 V :\equiv$ to every $v \in V$ there is $u \in U$ such that $v \leq u$.

Then $U \leq_1 V$ iff there is a mapping $\delta_U^V : V \to U$

such that $v \leq \delta_U^V(v)$.

If such a mapping exists, then it is unique.

Suppose that $\mathcal{U} \subseteq \mathcal{M}$ and $\mathcal{U} \leq_1 V$.

Lemma

V is a join of $\mathcal{U}$ iff the family $(\delta_U^V : U \in \mathcal{U})$ is separating:

for any $v_1, v_2 \in V$,

if $\delta_U^V(v_1) = \delta_U^V(v_2)$ for all $U \in \mathcal{U}$, then $v_1 = v_2$.

A *selector* for $\mathcal{U}$ – a function $\phi \in \prod \mathcal{U}$ such that

for all $U_1, U_2 \in \mathcal{U}$,

if $U_1 \leq_1 U_2$, then $\phi(U_1) = \delta_{U_1}^{U_2}(\phi(U_2))$;

equivalently,

if $U_1 \leq_1 U_2$, then $\phi(U_2) \leq \phi(U_1)$.

A *selector* for $\mathcal{U}$ – a function $\phi \in \prod \mathcal{U}$ such that
for all $U_1, U_2 \in \mathcal{U}$,
if $U_1 \leq_1 U_2$, then $\phi(U_1) = \delta_{U_1}^{U_2}(\phi(U_2))$;

equivalently,
if $U_1 \leq_1 U_2$, then $\phi(U_2) \leq \phi(U_1)$.

δ-product of $\mathcal{U} := \prod^{\delta} \mathcal{U} :=$ the set of all selectors.

A *selector* for $\mathcal{U}$ – a function $\phi \in \prod \mathcal{U}$ such that

for all $U_1, U_2 \in \mathcal{U}$,

if $U_1 \leq_1 U_2$, then $\phi(U_1) = \delta_{U_1}^{U_2}(\phi(U_2))$;

equivalently,

if $U_1 \leq_1 U_2$, then $\phi(U_2) \leq \phi(U_1)$.

δ-product of $\mathcal{U} := \prod^\delta \mathcal{U} :=$ the set of all selectors.

Example (still $\mathcal{U} \leq V$):

given $v \in V$, let $\phi_v(U) := \delta_U^V(v)$ for all $U \in \mathcal{U}$; then $\phi_v \in \prod^\delta \mathcal{U}$.

A *selector* for $\mathcal{U}$ – a function $\phi \in \prod \mathcal{U}$ such that
for all $U_1, U_2 \in \mathcal{U}$,
if $U_1 \leq_1 U_2$, then $\phi(U_1) = \delta_{U_1}^{U_2}(\phi(U_2))$;
equivalently,
if $U_1 \leq_1 U_2$, then $\phi(U_2) \leq \phi(U_1)$.

δ-product of $\mathcal{U}$:= $\prod^\delta \mathcal{U}$:= the set of all selectors.

Example (still $\mathcal{U} \leq V$):
given $v \in V$, let $\phi_v(U) := \delta_U^V(v)$ for all $U \in \mathcal{U}$; then $\phi_v \in \prod^\delta \mathcal{U}$.

Lemma

(a) For every $\phi \in \prod^\delta \mathcal{U}$, there is $v \in V$ such that $\phi = \phi_v$.

(b) V is a join of $\mathcal{U}$ iff such v is unique.