On the quasi-normality of commutative f-rings with bounded inversion

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Throughout, the term “ring" means a commutative ring with identity. An *ℓ-ring* is a ring A with a partial order $\leq$ such that, for every $a, b, c \in A$:

- $a + (b \vee c) = (a + b) \vee (a + c)$; and
- $ab \geq 0 \quad$ whenever $a \geq 0$ and $b \geq 0$.

An ℓ-ring is called an *f-ring* if

- $a(b \vee c) = (ab) \vee (ac)$ whenever $a \geq 0$.

The absolute value of $a \in A$ is the element

$$|a| = a \vee (-a).$$

An ideal $I \subseteq A$ is called an ℓ-ideal if

$$|a| \leq |b| \quad \text{and} \quad b \in I \quad \Longrightarrow \quad a \in I.$$

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In the article

S. Larson

A characterization of f-rings in which the sum of semiprime ℓ-ideals is semiprime and its consequences

Comm. Algebra **23** (1995), 5461-5481

Larson defines an f-ring to be quasi-normal if the sum of any two different minimal prime ℓ-ideals is either a maximal ℓ-ideal or the entire f-ring.

She then shows that, for Tychonoff space X, $C(\beta X)$ is quasi-normal if and only if $C(X)$ is quasi-normal and O^p is a prime ideal whenever M^p is hyper-real.

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If A is an f-ring, the bounded part of A, denoted A^*, is the subring

$$A^* = \{a \in A \mid |a| \leq n.1 \text{ for some } n \in \mathbb{N}\}.$$

If P is a prime ideal of A, the zero-component of P, denoted O_P, is the ideal

$$O_P = \{a \in A \mid ab = 0 \text{ for some } b \in A \smallsetminus P\}.$$

Denote by $[a]$ the principal ℓ-ideal of an f-ring A generated by $a \in A$, and recall that, for any $x \in A$,

$$x \in [a] \quad \Longleftrightarrow \quad |x| \leq r|a| \text{ for some } r \geq 0.$$

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B. Banaschewski

Functorial maximal spectra

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Banaschewski proves the following result:

> *For any f-ring A, the identical embedding* $A^* \to A$ *induces a homeomorphism* $\mathrm{Max}(A^*) \to \mathrm{Max}(A)$ *taking* P *to* $\{a \in A \mid [a] \cap A^* \subseteq P\}$.

Given a maximal ideal P of A^*, let us put

$$\tilde{P} = \{a \in A \mid [a] \cap A^* \subseteq P\}.$$

So for every maximal ideal M of A there is a unique maximal ideal M^* of A^* such that $M = \widetilde{M^*}$.

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An f-ring A has bounded inversion if any $a \in A$ with $a \geq 1$ is invertible. If A has bounded inversion, and

$$S = \{a \in A^* \mid a \text{ is invertible in } A\}$$

then $A = A^*[S^{-1}]$.

Lemma

Let A be an f-ring with bounded inversion, M be a maximal ideal in A, and M^* be the unique maximal ideal in A^* such that $M = \widetilde{M^*}$.

(a) *If M^* contains a unit of A, then $M^c \subset M^*$ (proper inclusion).*

(b) *If M^* contains no unit of A, then $M^c = M^*$.*

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Theorem

Let A be a reduced f-ring with bounded inversion. Suppose that the sum of two minimal prime ideals in A^* is a prime ideal if it is proper. Then A^* is quasi-normal if and only if A is quasi-normal and O_M is a prime ideal for every maximal ideal M of A for which M^* contains a unit of A.

Remark

The requirement that the sum of two minimal prime ideals in A^* be prime if it is proper forces the same for A.

- If P and Q are minimal prime ideals in A with $P + Q \neq A$, then P^c and Q^c are minimal prime ideals in A^* with $P^c + Q^c \neq A^*$.
- So $(P + Q)^c$ is prime, and if $ab \in P + Q$ for $a, b \in A$, then $\frac{a}{1+|a|} \cdot \frac{b}{1+|b|} \in (P + Q)^c$, so that we may assume $\frac{a}{1+|a|} \in (P + Q)^c$, which implies $a \in P + Q$, showing that $P + Q$ is prime.

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