Mixed algebras and their logics

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Mixed algebras

- A structure $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, f, g \rangle$ is called a PS-algebra if
  - B is a Boolean algebra.
  - f is a **p**ossibility operator on B, i.e. $f(0) = 0$ and $f(a+b) = f(a) + f(b)$.
  - g is a **s**ufficiency operator on B, i.e. $g(0) = 1$ and $g(a+b) = g(a) \cdot g(b)$.

  Define $h^*(a) \stackrel{\text{def}}{=} -h(a)$. Then,

  f is a possibility operator if and only if f^* is a sufficiency operator.
- The canonical frame $\mathrm{Cf}(\mathfrak{B})$ of $\mathfrak{B}$ is the structure $\langle X, R_f, R_g \rangle$ where $X = \mathrm{Ult}(B)$ and

$$\langle F, G \rangle \in R_f \iff G \subseteq f^{-1}(F),$$
$$\langle F, G \rangle \in R_g \iff F \cap g[G] \neq \varnothing.$$

- $\mathfrak{B}$ is called a mixed algebra if $R_g = R_f$.
- The class **MIA** of mixed algebras is not first order axiomatizable.

Weak mixed algebras

- Generalize **MIA** to the class **WMIA** of PS – algebras which satisfy $R_g \subseteq R_f$ in the PS – canonical frame $\langle \mathrm{Ult}(B), R_f, R_g \rangle$ of $\mathfrak{B}$, i.e. $R_f \cup -R_g = \mathrm{Ult}(B)^2$.
- If $\mathfrak{B} = \langle B, f, g \rangle$ is a PS – algebra then $\mathfrak{B}$ is in **WMIA** if and only if

$$(\forall a)[a \neq 0 \Rightarrow g(a) \leq f(a)]. \tag{1}$$

- Consider the mapping $u : B \to B$ defined by $u(a) = f^{\partial}(a) \cdot g(-a)$. Then, $\mathfrak{B}$ is a weak MIA if and only if for all $a \in B$

$$u(a) = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Each element of **WMIA** is a discriminator algebra, hence, **WMIA** is not an equational class.

$K^{\sim}$– algebras

- $\mathfrak{B}$ is called a $K^{\sim}$-algebra if and only if $R_f \cup -R_g$ is an equivalence relation.
- The class **KMIA** of $K^{\sim}$ – algebras is equational:

  $\mathfrak{B} \in \mathbf{KMIA}$ if and only if u is an S5 operator, i.e.

$$\begin{aligned} u(a) &\leq a, \\ u(a) &\leq u(u(a)), \\ a &\leq u(u^{\partial}(a)). \end{aligned}$$

- $\mathbf{Eq}(\mathbf{WMIA}) = \mathbf{KMIA}$.

Mixed frames

- If $\mathscr{X} = \langle X, R, S\rangle$ is a frame with two binary relations, its mixed complex algebra is the structure $\mathrm{Cm}(\mathscr{X}) = \langle 2^X, \langle R\rangle, [[S]]\rangle$, where

$$\langle R\rangle(A) \stackrel{\text{def}}{=} \{x : (\exists y)[xRy \text{ and } y \in A]\} = \{x : R(x) \cap A \neq \varnothing\} \quad \text{“Possibility”}$$
$$[[S]](A) \stackrel{\text{def}}{=} \{x : (\forall y)[y \in A \Rightarrow xSy]\} = \{x : A \subseteq S(x)\}. \quad \text{“Sufficiency”}$$

- $\mathscr{X}$ is called a MIA frame if $S = R$.
- $\mathscr{X}$ is called a weak MIA frame if $S \subseteq R$.
- $\mathscr{X}$ is called a $K^{\tilde{}}$ – frame if $R \cup -S$ is an equivalence relation.

Theorem 1.

1. The canonical frame of a weak mixed algebra ($K^{\tilde{}}$– algebra) is a weak MIA frame ($K^{\tilde{}}$ – frame).
2. The complex algebra of a weak MIA frame ($K^{\tilde{}}$ – frame) is in **WMIA** (**KMIA**).
3. Each weak MIA (KMIA) can be embedded into the complex algebra of its canonical frame.

The logics K and $K^\star$

- The logic K is a Boolean logic with a modal operator $\Box$ whose axioms are

$$\vdash \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi \tag{2}$$

$$\text{If } \vdash \varphi, \text{ then } \vdash \Box\varphi. \tag{3}$$

- Frame models are relational structures $\langle X, R, v \rangle$ in such a way that for a valuation $v : \mathrm{Fml}^K \to 2^X$ its action with respect to $\Box$ is given by

$$x \in v(\Box\varphi) \iff R(x) \subseteq v(\varphi).$$

- The logic $K^\star$ is a Boolean logic with a modal operator $\boxbar$ whose axioms are

$$\vdash \boxbar\neg(\varphi \to \psi) \to (\boxbar\neg\varphi \to \boxbar\neg\psi) \tag{4}$$

$$\text{If } \vdash \varphi, \text{ then } \vdash \boxbar\neg\varphi. \tag{5}$$

- Frame models are relational structures $\langle X, S, v \rangle$ in such a way that for a valuation $v : \mathrm{Fml}^* \to 2^X$ its action with respect to $\boxbar$ is given by

$$x \in v(\boxbar\varphi) \iff v(\varphi) \subseteq S(x),$$

- Correspondence:

$$\langle X, R, v \rangle, x \models \Box\varphi \iff \langle X, X^2 \setminus R, v \rangle, x \models \boxbar\neg\varphi.$$

The logic $K^{\sim}$ (Gargov et al. [3])

- Modal operators: $\Box$ (necessity) and $\boxminus$ (sufficiency).
- In addition to axioms for $\Box$ and $\boxminus$, the auxiliary operator $[U]\varphi \stackrel{\text{def}}{=} \Box\varphi \wedge \boxminus\neg\varphi$ is an S5 modality, i.e.

$$[U]\varphi \to \varphi,\ [U]\varphi \to [U][U]\varphi,\ \varphi \to [U]\langle U\rangle\varphi.$$

- Frame models have the form $M = \langle X, R, S, v\rangle$ where $S \subseteq R$. With respect to the modal operators, a valuation v acts as follows:

$$x \in v(\Box\varphi) \Longleftrightarrow R(x) \subseteq v(\varphi), \qquad \text{Necessity } [R]$$
$$x \in v(\boxminus\varphi) \Longleftrightarrow v(\varphi) \subseteq S(x). \qquad \text{Sufficiency } [[S]]$$

A model $\langle X, R, S, v\rangle$ of $K^{\sim}$ is called special if $R = S$

Theorem 2. Gargov et al. [3]

1. $K^{\sim}$ is sound and complete with respect to its class of frame models.
2. Each model of $K^{\sim}$ is modally equivalent to a special model.

Algebraic models of $K^{\sim}$

Theorem 3. For all $\varphi \in \mathrm{Fml}^{K^{\sim}}$, $K^{\sim} \models \varphi$ if and only if **KMIA** $\models \varphi$.

Theorem 4. If φ is a formula in $K^{\sim}$, then $K^{\sim} \vdash \varphi$ if and only if **KMIA** $\models \varphi$.

Putting together the completeness results and Theorem 2 we have

Theorem 5.Let $\mathfrak{B} = \langle B, f, g \rangle \in$ **KMIA**. Then, there is some frame $\langle X, R \rangle$ such that $\langle B, f, g \rangle$ and a subalgebra of $\langle 2^X, \langle R \rangle, [[R]] \rangle$ satisfy the same equations.

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Köszönöm
Mnogo blagodarya
Děkuji
Dziękuję
Grazie
Thank you
Danke
Merci