Unification of terms and language expansions

in Heyting algebras and commutative residuated lattices

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joint work with

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Examples of frontal Heyting algebras, in particular Heyting algebras with the successor, γ and G operations as well as expansions of some commutative integral residuated lattices with some successors are considered.

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A substitution $\sigma : \{x_1, \ldots, x_n\} \to T(\underline{y})$ is an $E-$ *unifier* for t_1, t_2 if

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Given two E-unifiers σ, $\tau : \underline{x} \to T(\underline{y})$ for t_1, t_2,
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A *mgu, most general unifier* for t_1, t_2 in E is a $E-$unifier σ for t_1, t_2 such that σ is more general then any E-unifier for t_1, t_2.

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If unification in E is unitary (or finitary) and decidable, then there are applications of some deduction technique to Automated Theorem Provers, industrial databases, Description Logic etc.
Not applicable if unification in E is infinitary or nullary.

Applications in logic: admissibility of inference rules.

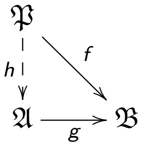
semigroups	infinitary	Plotkin 1972
commutat. semigroups	finitary	Livesey, Siekmann 1976
groups	infinitary	Lawrence 1989
Abelian groups	finitary	Lankford 1979
rings	infinitary	Lawrence 1987
commutat. rings	nullary or infinitary	Burris, Lawrence 1989
lattices (distr.latt.)	nullary	Willard 1991
semilattices	finitary	Livesey, Siekmann 1976
Boolean algebras	unitary	Biitner, Simonis 1987
discriminator algebras	unitary	Burris 1987
Heyting algebras	finitary	Ghilardi 1999
closure (interior) alg.	finitary	Ghilardi 2000
equivalential algebras	unitary	Wroński 2005
q-linear closure (int.) a.	unitary	Dzik, Wojtylak 2011
Fregean varieties	unitary	Slomczynska 2011
MV- algebras (Łukas.)	nullary	Marra, Spada 2011
var. dis. pseudocpl. latt.	nullary	Cabrer 2013

See Stan Burris http://www.math.uwaterloo.ca/ ∼ snburris/

Algebraic approach by Ghilardi (1999)

An algebra $\mathfrak{B} \in \mathcal{V}_E$ is *finitely presented*, if there is a finite set of variables, $x_1, \ldots, x_k = \underline{x}$ and a finite set S of equations of terms with variables in $\underline{x}$ such that $\mathfrak{B}$ is isomorphic to a quotient algebra $\mathcal{F}_E(\underline{x})/_\sim$, where $\sim$ is a congruence defined as follows :
$t_1 \sim t_2$ iff $S \vdash_E t_1 = t_2$

An algebra $\mathfrak{P}$ in a variety $\mathcal{V}$ is *projective* in $\mathcal{V}$ if for every $\mathfrak{A}$, $\mathfrak{B}$ of $\mathcal{V}$ and homomorphisms $f : \mathfrak{P} \to \mathfrak{B}$, $g : \mathfrak{A} \to \mathfrak{B}$ (where g is epi) there is a $h : \mathfrak{P} \to \mathfrak{A}$ such that the following diagram commutes



$\mathfrak{P}$ is projective in $\mathcal{V}$ iff $\mathfrak{P}$ is a *retract of a free algebra* in $\mathcal{V}$, i.e. there are q and m such that

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A *unifier (a solution)* for $\mathfrak{A}$ is a pair given by: a projective algebra $\mathcal{P}$ and a homomorphism $u : \mathfrak{A} \to \mathcal{P}$.

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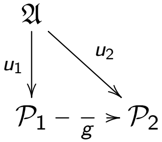
Given two unifiers u_1 and u_2 for $\mathfrak{A}$, $u_1 : \mathfrak{A} \to \mathcal{P}_1$ is *more general* then $u_2 : \mathfrak{A} \to \mathcal{P}_2$, $u_2 \preceq u_1$, if there is a homomorphism g such that the following diagram commutes:



Unification types in symbolic and algebraic approach coincide.

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Theorem (Ghilardi 2003) Unification in L is filtering iff the *direct product of two finitely presented and projective algebras from $\mathcal{V}_L$ is finitely presented and projective*.

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for any $c \in \mathrm{Cen}(L)$, θ_c and $\theta_{\overline{c}}$ form a factor congruence pair of L;
Conversely, if θ_1 and θ_2 are factor congruences of a bounded lattice L: $L \cong L/\theta_1 \times L/\theta_2$, then there exists a $c \in \mathrm{Cen}(L)$ such that $\theta_1 = \theta_c$, $\theta_2 = \theta_{\overline{c}}$.

An algebra $\mathcal{A} = (L, \wedge, \vee, 0, 1, F)$ is called an *l-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, and any n-ary term $f\colon L^n \to L$ of it is *centre-preserving,* that is, for every $c \in \mathrm{Cen}(L)$,

$$(x_i, y_i) \in \theta_c,\ i = 1, ..., n \text{ implies } (f(x_1, ..., x_n), f(y_1, ..., y_n)) \in \theta_c.$$

Examples of l-algebras: bounded lattices, p-algebras, ortholattices, Heyting algebras etc., also residuated lattices (we will show it)
- any l-algebra is congruence distributive,
- the factor congruences of an l-algebra and of its underlying lattice L coincide.

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Theorem *Let $\mathcal{V}$ be a 1-regular variety of l-algebras satisfying the conditions (A), (B) and (C). If $\mathcal{A}, \mathcal{B} \in \mathcal{V}$ are finitely presented projective algebras, then $\mathcal{A} \times \mathcal{B}$ is a fin. presen. proj. algebra of $\mathcal{V}$.*

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Corollary: *(A), (B), (C) $\Rightarrow$ unification in $\mathcal{V}$ is either unitary or null*

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An algebra $\mathcal{L} = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an *integral bounded commutative residuated lattice*, or *bounded residuated* lattice, if
(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \odot)$ is a commutative monoid with unit element 1;
(3) $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z$, for all $x, y, z \in L$.

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Theorem *Any bounded residuated lattice is an l-algebra satisfying condition (A).*

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Theorem. *Let $\mathcal{V}$ be a variety of (commutative integral bounded) residuated lattices. Then the following are equivalent:*

(i) unification in $\mathcal{V}$ is filtering and the identity $\neg x \odot \neg x = \neg x$ *holds in the variety* $\mathcal{V}$,

(ii) $\mathcal{V}$ *is Stonean.*

(iii) the subdirectly irreducible members of $\mathcal{V}$ *have no zero divisors.*

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adding (removing) connectives can change unification type

Heying algebras with compatible operations.

Heyting algebras with compatible operations.

Adding new connectives to (intuitionistic) logic, in a natural way, has been studied by several authors, e.g. Esakia, Gabbay, Caicedo,

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An algebra H, a function (operation) $f : H^n \to H$ is *compatible with* H if it is compatible with every congruence θ of H, that is: $x_i \theta y_i,\ i \leq n \quad \Rightarrow\ f(x_1, \ldots, x_n)\ \theta\ f(y_1, \ldots, y_n)$, for every θ.

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In a Heyting algebra H, an unary f is compatible with H iff
$f(x) \wedge a = f(x \wedge a) \wedge a$

A compatible operation is new if it can not be defined by the basic operations.

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(f1) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$,

(f2) $x \leq \tau(x)$,

(f3) $\tau(x) \leq y \vee (y \rightarrow x)$.

Any frontal operation τ is a compatible operation, by (f1) and (f2)

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For a finite chain $H_n = \{0, 1, \dots, n\}$ of $(n+1)$ elements treated as a Heyting algebra with the natural order, i.e. $\bot = 0, \top = n$, $\wedge = min$, $\vee = max$, $a \to b = b$ if $a > b$, or $:= 1$ otherwise, we have:

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S is defined on any Heyting algebra without inf. descending chains.

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Theorem

Let $\mathcal{V}^*$ be an expansion of algebras of any variety $\mathcal{V}$ of Heyting algebras with new compatible operations. If $\neg\neg x \vee \neg x = 1$ holds in $\mathcal{V}$, then unification in $\mathcal{V}^*$ is filtering, hence it is unitary or nullary.

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Corollary

Let L^* be any expansion of a logic L with new compatible connectives and $INT \subseteq L$. If $\neg\neg A \vee \neg A \in L$, then unification in L^* is filtering, hence it is unitary or nullary.
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Theorem

Any extension of intuitionistic propositional logic INT with compatible connectives can not have unitary unification. Any expansion of the variety $\mathcal{V}$ of all Heyting algebras with new compatible operations can not have unitary unification.

Hence one can not "improve" unification of intuitionistic logic INT by adding compatible connectives.

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Unification in an expansion of the variety $\mathcal{LC}$ of linearly ordered Heyting algebras of order $\leq n$ with the successor operation S remains unitary.
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If the above operations are added to intuitionistic logic (or to the variety of all Heyting algebras) then unification can not be unitary, but it is not known what is the unification type in each case.

Commutatie Residuated lattices with compatible operations.

Corollary

Unification in commutative integral residuated lattices satisfying the Stone identity $\neg\neg x \vee \neg x = 1$ *with additional compatible operations is filtering, that is, unitary or nullary.*

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Unification in commutative integral residuated lattices satisfying the Stone identity $\neg\neg x \vee \neg x = 1$ *with additional compatible operations* S_n*, if defined, is filtering, that is, unitary or nullary.*

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