On the lattice of congruences on a fruitful semigroup

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Outline

Definition

By a **semigroup** we shall mean a non-empty set S together with an associative (binary) operation.

Definition

Let S be a semigroup. The set

$$E_S = \{e \in S : ee = e\}$$

is called the set of **idempotents** of S.

NOTATION. Let ρ be an equivalence relation on a semigroup S. For any $a \in S$, denote the ρ-class containing a by $a\rho$, and put

$$S/\rho = \{a\rho : a \in S\}.$$

Definition

An equivalence relation ρ on a semigroup S is said to be a **congruence** if for all $a, b \in S$, the algebraic product $a\rho \cdot b\rho$ is always contained in a single ρ-class of S; namely, the ρ-class $(ab)\rho$.

- Let ρ be a congruence on a semigroup S. Then the quotient space $S/\rho = \{a\rho : a \in S\}$ is a semigroup with respect to the multiplication

$$(a\rho)(b\rho) = (ab)\rho.$$

Denote the natural morphism from S onto S/ρ by $\rho^\natural$, that is, $a\rho^\natural = a\rho$ $(a \in S)$.

Fact

Let ρ be a congruence on a semigroup S and let $a \in S$. Suppose also that the element $a\rho$ is an idempotent of S/ρ. Then the ρ-class $a\rho$ is a ***subsemigroup*** *of S.*

Definition

By a **topological semigroup** we shall mean an algebraic semigroup S which is simultaneously a Hausdorff topological space, and whose semigroup operation $S \times S \to S$ is continuous (where $S \times S$ is the product topology).
If in addition, S is a compact space, then S is a **compact semigroup**.

Definition

Let S be a topological semigroup. A congruence on S is called **topological** if S/ρ is a topological semigroup with respect to the quotient topology

$$\mathcal{O}_{S/\rho} = \{A \subseteq S/\rho : A{\rho^{\natural}}^{-1} \in \mathcal{O}_S\}.$$

The following simple (but important) result characterizes topological congruences in the class of compact semigroups [10] (Gigoń).

Fact

A congruence on a compact semigroup S is ***topological*** *if and only if it is* ***closed*** *in the product topology $S \times S$.*
Hence the set $\mathcal{TC}(S)$ of all topological congruences on a compact semigroup S forms a ***complete lattice****.*

NOTATION. Let S be a semigroup. Denote by $\mathcal{C}(S)$ the complete lattice of congruences on S.

Definition

A semigroup S is said to be **regular** if for every a in S there is $x \in S$ such that $a = axa$.

REMARK. Regular semigroups were introduced by J. A. Green in his influential 1951 paper *On the structure of semigroups* [16]; this was also the paper in which (the so-called now) Green's relations were introduced. The concept of regularity in a semigroup was adapted from an analogous condition for rings, already considered by J. von Neumann [23].
Regular semigroups are one of the most-studied classes of semigroups, and their structure is particularly amenable to study via Green's relations.

The following definitions were introduced by Edwards in [2].

Definition

A congruence ρ on a semigroup S is said to be **idempotent-surjective** if every idempotent congruence ρ-class A of S/ρ contains an idempotent of S, that is, $A \cap E_S \neq \emptyset$.

Definition

A semigroup S is called **idempotent-surjective** if each of its congruences is idempotent-surjective.

The following famous lemma is due to Lallement (1966).

Lemma

Regular semigroups are idempotent-surjective.

The following notion was introduced by Edwards in [2] (1983).

Definition

A semigroup S is said to be **eventually regular** if each of its elements has a regular power, that is, for every a in S there exists a positive integer n and an element x of S such that $a^n = a^n x a^n$ (i.e., the element a^n is regular).

REMARK. Eventually regular semigroups are known also under the names: π**-regular semigroups** and **quasi-regular semigroups**.

The following has been proved by Edwards [2] (1983).

Fact

Eventually regular semigroups are idempotent-surjective.

The following semigroups are examples of eventually regular semigroups:

- **finite semigroups**;
- **periodic semigroups** (that is, every element has an idempotent power);
- **epigroups** or **group-bound semigroups** or **completely π-regular semigroups** or **quasi-periodic semigroups** (that is, every element has a power that belongs to a subgroup).

Consider the following congruence on a semigroup S:

$$\theta_{m,n} = \{(a,b) \in \ S \times S : (\forall x \in S^m, y \in S^n)(xay = xby)\},$$

where $m, n \in \{0,1,2,\dots\}$, $S^1 = S$ and S^0 denotes the set containing only the empty word (hence $\theta_{0,0} = 1_S$).

REMARK. Note that if a semigroup S has an identity, then $\theta_{m,n} = 1_S$ for all $m, n \in \{0,1,2,\dots\}$.

The following concept was introduced by Kopamu in [18] (1996).

Definition

A semigroup S is called **structurally regular** if $S/\theta_{m,n}$ is a regular semigroup for some non-negative integers m, n.

REMARK.

- Any regular semigroup S is structurally regular, as $S/\theta_{0,0} \cong S$.
- One can introduce the notion of a **structurally eventually regular semigroup** in a similar way.
- Any eventually regular semigroup S is structurally eventually regular.

The following are due to Kopamu [18] (1996).

Fact

Structurally regular semigroups are idempotent-surjective.

Fact

The classes of eventually regular semigroups and structurally regular semigroups are incomparable, that is, neither contains the other.

REMARK. In 1997 [17] Higgins constructed an idempotent-surjective semigroup with **identity** which is not eventually regular (so this semigroup is not structurally eventually regular too).

Let S be a semigroup and let $a \in S$. The set

$$W_S(a) = \{x \in S : x = xax\}$$

is said to be the **set of all weak inverses** of a. Notice that in such a case, $xa, ax \in E_S$.

Definition

We say that a semigroup S is E**-inversive** if $W_S(a) \neq \emptyset$ for every a in S.

REMARK.

- The notion of a E-inversive semigroup was introduced in 1955 by Thierrin [21] (but only in the 90s of the 20th century, began to appear interesting results on this semigroup).
  Recall that this class is very extensive. In fact, **almost all** semigroups studied in the literature are E-inversive.
- Note that idempotent-surjective semigroups are E-inversive [17] (Higgins 1997).
- Not every E-inversive semigroup is idempotent-surjective. Indeed, consider the semigroup $S = \{0, 1, 2, \dots\}$ with multiplication. It is clear that the partition $\{\{0, 1\}, \{2, 3, 4, \dots\}\}$ induces a congruence on this semigroup, say ρ, and that the class $\{2, 3, 4, \dots\}$ is an idempotent of S/ρ but this class is idempotent-free.

In semigroup theory by an **'important' congruence** we mean such a congruence ρ on some semigroup S that the semigroup S/ρ belongs to some well-known (and well-studied) class of semigroups. For example, a congruence ρ on a semigroup S is said to be:

- a **left zero** congruence if S/ρ is a left zero semigroup (i.e., it satisfies the identity $xy = x$).
- a **right zero** congruence if S/ρ is a right zero semigroup (i.e., it satisfies the identity $xy = y$).
- a **matrix** congruence if S/ρ is a rectangular band (that is, the direct product of a left zero semigroup and a right zero semigroup).
- a **semilattice** congruence if S/ρ is a semilattice (i.e., an idempotent commutative semigroup).
- a **semilattice of groups** congruence if S/ρ is a semilattice of groups (a semigroup A is called a **semilattice of groups** if there is a semilattice congruence ρ on A such that every ρ-class of A is a group).
- **completely simple** if S/ρ is a completely simple semigroup (i.e., a matrix of groups).

REMARK. In my previous articles, certain congruences in some classes of semigroups have the following property: **every idempotent congruence class is an *E*-inversive semigroup.** For example:

- all **semilattice of groups congruences** on an arbitrary eventually regular semigroup have this property [7].
- all **matrix congruences** on an arbitrary *E*-inversive semigroup have this property [5, 9].
- all **completely simple congruences** on an arbitrary *E*-inversive semigroup have this property [8].

This motivates me to introduce the following definitions.

Definition

A congruence ρ on a semigroup S is said to be **fruitful** if the condition $a\rho$ ($a \in S$) is an idempotent of S/ρ implies that the ρ-class $a\rho$ is an E-inversive semigroup.

Definition

A semigroup is called **fruitful** if each of its congruences is fruitful.

Definition

A topological semigroup is called **fruitful** if each of its topological congruences is fruitful.

REMARK. The following semigroups are examples of fruitful semigroups:

- structurally eventually regular semigroups [11].
- structurally regular semigroups.
- eventually regular semigroups.
- compact semigroups [11].

Definition

A congruence ρ on a semigroup S is **idempotent-separating** if every ρ-class of S contains **at most one** idempotent.

REMARK. Recall that Edwards showed in 1989 that an arbitrary idempotent-surjective semigroup has a greatest idempotent-separating congruence, say μ [4].

In [12] I have generalized the description of the greatest idempotent-separating congruence μ on an arbitrary eventually regular semigroup (see Luo and Li 2007 [19]) to an arbitrary fruitful semigroup which is not a compact semigroup (the below relation is not closed in general; indeed, one can consider the compact semigroup $[0, 1]$).

Theorem

Let S be a fruitful semigroup which is not compact. Then $(a, b) \in \mu$ if and only if

$$\begin{cases} \big(\forall a^* \in W_S(a)\ \exists\, b^* \in W_S(b)\big)(aa^* = bb^*\ \&\ a^*a = b^*b) \\ \big(\forall b^* \in W_S(b)\ \exists\, a^* \in W_S(a)\big)(aa^* = bb^*\ \&\ a^*a = b^*b). \end{cases}$$

Definition

Let ρ be a congruence on a semigroup S with $E_S \neq \emptyset$. The relation

$$\mathrm{tr}(\rho) = \rho \cap (E_S \times E_S)$$

is called the **trace** of ρ.

Definition

Let S be a semigroup with $E_S \neq \emptyset$. Put

$$\theta = \{(\rho_1, \rho_2) \in \mathcal{C}(S) \times \mathcal{C}(S) : \mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)\}.$$

REMARK. Suppose that S is a fruitful semigroup. Note that θ is an equivalence relation on $\mathcal{C}(S)$. Let $\rho \in \mathcal{C}(S)$. Then clearly the intersection of all elements of $\rho\theta$ belongs to the set $\rho\theta$, so it is a least element of $\rho\theta$. Denote it by $0(\rho)$.
Recall from [11] that the class of fruitful semigroups is closed under taking homomorphic images.
Observe that

$$\mu(\rho) = \{(a, b) \in S \times S : (a\rho, b\rho) \in \mu\}$$

is a congruence on S, $\rho \subseteq \mu(\rho)$ and $\mathrm{tr}(\rho) = \mathrm{tr}(\mu(\rho))$.

The following (see [12]) generalizes the result of Edwards (1985) on the lattice of congruence on eventually regular semigroups [3].

Theorem

Let S be a fruitful semigroup (which is not compact). Then the following statements hold:
(a) θ is a complete congruence on $\mathcal{C}(S)$;
(b) for every $\rho \in \mathcal{C}(S)$, $\rho\theta = [0(\rho), \mu(\rho)]$ is a complete sublattice of $\mathcal{C}(S)$.

Let S be a semigroup (with $E_S \neq \emptyset$) from some special class of semigroups (for instance, from the class of regular semigroups). Denote the complete lattice of equivalences on E_S by $\mathcal{E}(E_S)$, and consider the following problem.

Problem

Is the map $\bar{\theta} : \mathcal{C}(S) \to \mathcal{E}(E_S)$, where $\rho\bar{\theta} = tr(\rho)$ for every $\rho \in \mathcal{C}(S)$, a complete lattice homomorphism between the complete lattices $\mathcal{C}(S)$ and $\mathcal{E}(E_S)$?

REMARK. Let $\{\rho_i : i \in I\}$ be a non-empty family of congruences on a fruitful semigroup S. It is very well-known that

$$\bigvee\{\rho_i : i \in I\} = \left(\bigcup\{\rho_i : i \in I\}\right)^t,$$

where the symbol α^t denotes the transitive closure of the relation α on S. The only non-obvious thing in the proof that the above function $\bar{\theta}$ is a complete lattice homomorphism between the complete lattices $\mathcal{C}(S)$ and $\mathcal{E}(E_S)$ is to prove that if

$$(e, f) \in \rho = \bigvee\{\rho_i : i \in I\},$$

where $e, f \in E_S$, i.e., if there exist $\rho_1, \rho_2, \ldots, \rho_n \in \{\rho_i : i \in I\}$ and the elements $a_1, a_2, \ldots, a_{n-1} \in S$ such that $e\, \rho_1\, a_1\, \rho_2\, a_2\, \rho_3 \ldots \rho_{n-1}\, a_{n-1}\, \rho_n\, f$, then

$$e\, \rho_1\, e_1\, \rho_2\, e_2\, \rho_3 \ldots \rho_{m-1}\, e_{m-1}\, \rho_m\, f$$

for some $e_1, e_2, \ldots, e_{m-1} \in E_S$.

We may assume without lost of generality that $I = \{1, 2, \ldots, n\}$. Also, if we denote the above relation by ϱ, that is, $\varrho = \bigvee\{\mathrm{tr}(\rho_i) : i \in I\} = \left(\bigcup\{\mathrm{tr}(\rho_i) : i \in I\}\right)^t$, then it is sufficient to show that $(e, f) \in \rho$ implies $(e, f) \in \varrho$.

Pastijn and Petrich solved this problem (in positive) for **orthodox semigroups** (i.e., for regular semigroups in which the set of idempotents forms a semigroup) and for **epigroups** [20]. They also put in [20] the above question for **regular semigroups** (the famous problem of Pastijn and Petrich). This problem has been solved by Trotter [22] (1986). Ten years later Auinger and Hall solved this problem for **eventually regular semigroups with some additional property**, see [1], and put there the above question for **eventually regular semigroups.** Recently, I have solved this problem for fruitful semigroups in [15] (and so the problem of Auinger and Hall has a positive solution). Namely, the following theorem is valid.

Theorem

If S is a fruitful semigroup (which is not compact), then the map $\bar{\theta} : \mathcal{C}(S) \to \mathcal{E}(E_S)$, where $\rho\bar{\theta} = tr(\rho)$ for every $\rho \in \mathcal{C}(S)$, is a complete lattice homomorphism between the complete lattices $\mathcal{C}(S)$ and $\mathcal{E}(E_S)$ which induces the complete congruence θ (see the above theorem).

COMPACT CASE. In the compact case there is a problem with joins. Namely, let ρ_1, ρ_2 be topological congruences (equivalently, closed congruences) on a compact semigroup. Then the join $\rho_1 \vee \rho_2 = (\rho_1 \cup \rho_2)^t$ is not always a closed relation.

K. AUINGER, T. E. HALL, *Representations of semigroups by transformations and the congruence lattice of an eventually regular semigroup*, Int. J. Algebra Comput. **6(6)** (1996), 655–685.

P. M. EDWARDS, *Eventually regular semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23–38.

P. M. EDWARDS, *On the lattice of congruences on an eventually regular semigroup*, J. Aust. Math. Soc. **38A** (1985), 281–286.

P. M. EDWARDS, *Maximizing a congruence with respect to its partition of idempotents*, Semigroup Forum **39** (1989), 257–262.

R. S. GIGOŃ, *Some results on E-inversive semigroups*, Quasigroups and Related Systems **20(1)** (2012), 53–60.

R. S. GIGOŃ, *Rectangular group congruences on a semigroup*, Semigroup Forum **87(1)** (2013), 120–128.

- R. S. GIGOŃ, *Certain congruences on eventually regular semigroups I*, Studia Scientiarum Mathematicarum Hungarica **52(4)** (2015), 434–449.
- R. S. GIGOŃ, *Completely simple congruences on E-inversive semigroups*, Journal of Algebra and Its Applications **15(6)** (2016), (17 pages).
- R. S. GIGOŃ, *Matrix congruences on E-inversive semigroups* (to appear in J. Aust. Math. Soc.).
- R. S. GIGOŃ, *Topological-congruence-free compact semigroups* (to appear in Topology and its Applications).
- R. S. GIGOŃ, *The concept of a fruitful congruence and a fruitful semigroup* (submitted to Semigroup Forum).
- R. S. GIGOŃ, *Description of the maximum idempotent-separating congruence on a fruitful semigroup with application to its lattice of congruences* (submitted to Semigroup Forum).

R. S. GIGOŃ, *Certain fundamental congruences on a fruitful semigroup* (submitted to Semigroup Forum).

R. S. GIGOŃ, *Fruitful congruences and Green's relations on semigroups* (submitted to Bull. Austral. Math. Soc.).

R. S. GIGOŃ, *Representations of semigroups by transformations and the congruence lattice of a fruitful semigroup* (submitted to Proceedings of the Royal Society of Edinburgh: Section A (Mathematics)).

J. A. GREEN, *On the structure of semigroups*, Annals of Mathematics. Second Series. Annals of Mathematics **54(1)** (1951), 163–172.

P. M. HIGGINS, *The converse of Lallement's lemma*, In: Semigroups and Applications (St Andrews, 1997), pp. 78–86, World Scientific, River Edge, NJ, 1998.

- S. J. L. KOPAMU, *The concept of structural regularity*, Port. Math. **53** (1996), 435–456.
- Y. LUO AND X. LI, *The Maximum Idempotent-Separating Congruence on Eventually Regular Semigroups*, Semigroup Forum **74** (2007), 306–317.
- F. PASTIJN, M. PETRICH, *Congruences on regular semigroups*, Trans. Amer. Math. Soc. **295** (1986), 607–633.
- G. THIERRIN, *Demi-groupes inverses et rectangulaires*, Bull. Cl. Sci. Acad. Roy. Belgique **41** (1955) 83-92.
- P. G. TROTTER, *On a problem of Pastijn and Petrich*, Semigroup Forum **34(1)** (1986), 249–252.
- J. VON NEUMANN, *On Regular Rings*, Proc. Natl. Acad. Sci. USA, **22(12)** (1936), 707–712.