The Krohn–Rhodes complexity of the flow semigroup of finite digraphs

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Joint work with Chrystopher L. Nehaniv and Károly Podoski
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Rhodes's conjecture

Conjecture (Rhodes)

Let Γ be a strongly connected, antisymmetric digraph on n points, and let S_Γ be its flow semigroup. Then

- *The Krohn–Rhodes complexity of S_Γ is $n-2$,*
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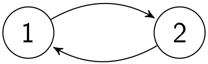
Graphs and digraphs

- *Digraph*: $\Gamma = (V, E)$, with *vertices* V, *edges* $E \subseteq V \times V$.
- *Reverse edge* to $e = (u, v) = uv \in E$ is $\bar{e} = (v, u) = vu$.
- *(Undirected) Graph*: E is symmetric.
- No self-loops: $(u, u) \notin E$.



Elementary collapsings

Digraph $\Gamma = (V, E)$

Elementary collapsing

edge $uv \longrightarrow$ function $e_{uv} \colon V \to V$

$$e_{uv}(x) = x \cdot e_{uv} = \begin{cases} v & \text{if } x = u, \\ x & \text{otherwise.} \end{cases}$$

(acting on the right)

Example

$$\begin{aligned} e_{12} \colon 1 &\to 2 \\ 2 &\to 2 \\ 3 &\to 3 \end{aligned}$$

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Example

e_{12}

Motivation: Biochemical Reactions

Biochemical transitions are modelled as products of commuting elementary collapsings, $f = \prod e_{ab}$, where e_{uv} and e_{vw} do not both occur among the e_{ab} for any u, v, and w.

Flow semigroup

Definition

Semigroup of flows: transformation semigroup acting on V generated by the e_{uv} $(uv \in E)$.

$$S_\Gamma = \langle e_{uv} \in V^V \mid (u, v) \text{ is an edge of } \Gamma \rangle.$$

Example (Γ is the 3-cycle with edges $(1,2),(2,3),(3,1)$)

compose functions from left to right

$$\begin{aligned} f = e_{23}e_{12}e_{31} : 1 &\to 1 \to 2 \to 2 \\ 2 &\to 3 \to 3 \to 1 \\ 3 &\to 3 \to 3 \to 1 \end{aligned}$$

Motivation

S_Γ is an invariant for digraphs, and a complete invariant on graphs.

Not complete invariant on digraphs

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Now $f^2 = e_{32}$ corresponds to the edge $(3,2) \notin \Gamma$.
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Similarly, one can 'reverse' any edge in a directed cycle.

Strongly connected digraphs

Proposition (Definition)

every edge is in a directed cycle (and connected).

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Digraph Γ. If uv is an edge in a directed cycle in Γ, then $S_\Gamma = S_{\Gamma \cup \{vu\}}$.

$\Longrightarrow$ Consider undirected graphs instead of strongly connected digraphs.

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What (undirected) graphs can we obtain? Any connected ones.

Antisymmetric graphs

Definition

A digraph is antisymmetric $\iff$ at most one of uv and vu is edge.

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Corollary

Enough to consider 2-edge connected (undirected) graphs.

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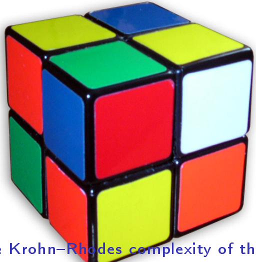
Wreath product

Definition (Wreath product of transformation semigroups)

(Y, T): T acting on Y
(X, S): S acting on X,
$T^X = \{f : X \to T\}$,
$(Y, T) \wr (X, S) = T^X \rtimes S$ with action on $Y \times X$ as

$$(y, x) \cdot (f, s) = (y \cdot f(x), x \cdot s).$$

Example (subgroup of $\mathbb{Z}_3 \wr S_8$)



Wreath product decompositions

Proposition (Associativity)

$\Big((X_3, S_3) \wr (X_2, S_2) \Big) \wr (X_1, S_1) \simeq (X_3, S_3) \wr \Big((X_2, S_2) \wr (X_1, S_1) \Big)$

Definition (Divisor)

$(X, S) \mid (Y, T) \iff$
$(X, S) \simeq$ homomorphic image of subsemigroup of (Y, T).

Theorem (Krohn–Rhodes decomposition)

$$(X, S) \mid (X_n, S_n) \wr (X_{n-1}, S_{n-1}) \wr \cdots \wr (X_1, S_1),$$

where (X_i, S_i) is either a simple group or the 'flip-flop'

$$(\{a, b\}, U_3) = (\{a, b\}, \{C_a, C_b, Id\}).$$

Motivation

Automata theory, simulating by cascade of automata.

Wreath product decompositions

Example (Groups)

G group, $N \lhd G \implies (G, G) \mid (N, N) \wr (G/N, G/N)$.
$G_1, \dots, G_n$ are the simple factors in order $\implies$ (Jordan–Hölder)

$$(G, G) \mid (G_n, G_n) \wr \cdots \wr (G_1, G_1).$$

Example

Γ is the triangle graph $\implies$

$$S_\Gamma \mid (\{a, b\}, U_3) \wr (\{a, b\}, \mathbb{Z}_2) \wr (\{a, b\}, U_3) \wr (\{a, b\}, U_3).$$

Krohn–Rhodes complexity

Proposition (Definition)

$$S \mid (\{a, b\}, U_3) \wr \cdots \wr (\{a, b\}, U_3)$$

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every maximal subgroup of S has exactly one element

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Definition (Krohn–Rhodes complexity $\#_G(S)$)

For arbitrary S the smallest non-negative integer n such that

$$S \mid C_n \wr G_n \wr C_{n-1} \wr \cdots \wr C_1 \wr G_1 \wr C_0,$$

where $G_1, \ldots, G_n$ are finite groups,
$C_0, \ldots, C_n$ are finite combinatorial semigroups

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Example

$\#_G(G) = 1$,
Γ is the triangle graph $\Longrightarrow \#_G(S_\Gamma) = 1$.

Krohn–Rhodes complexity, properties

- $S \mid T \Longrightarrow \#_G(S) \leq \#_G(T)$
- $\#_G(S \times T) \leq \max(\#_G(S), \#_G(T))$
- $\#_G(S \wr T) \leq \#_G(S) + \#_G(T)$
- $\#_G(F_n) = n - 1$
- $\#_G((X, S)) \leq |X| - 1$
- not known if it is decidable for arbitrary semigroups (decidable e.g. for **DS**, i.e. $((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega$)
- etc.

Conjecture (Rhodes)

Let Γ be a 2-edge connected (undirected) graph on n points, and let S_Γ be its flow semigroup. Then $\#_G(S_\Gamma) = n - 2$.

$\#_G(S_\Gamma) \leq n - 2$ is easy.
Problem. Much less is known about lower bounds.

Lower bounds

Lemma (Rhodes–Tilson)

If S is a noncombinatorial T_1-semigroup, then

$$\#_G(EG(S)) < \#_G(S),$$

where $EG(S)$ is the subsemigroup generated by all idempotents.

Idea of proving $n - 2 \leq \#_G(S_\Gamma)$

S_Γ (not T_1)

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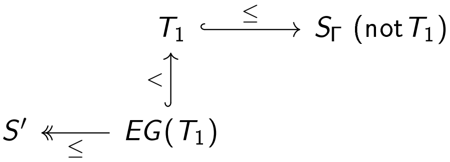

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$$\begin{array}{ccccccc}
 & & & & T_1 & \xhookrightarrow{\leq} & S_\Gamma \ (\text{not } T_1) \\
 & & & & {\scriptstyle <}\big\uparrow & & \\
S_{K_{n-1}} & \xhookrightarrow{\leq} & S' & \underset{\leq}{\twoheadleftarrow} & EG(T_1) & &
\end{array}$$

Finally...

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Let Γ be a 2-edge connected graph on n points,
and let S_Γ be its flow semigroup. Then $\#_G(S_\Gamma) = n - 2$.

Finally. . .

2-vertex connected $\iff$ does not disconnect by removing a vertex.

Theorem (Horváth, Nehaniv, Podoski)

Let Γ be a 2-vertex connected graph on n points, and let S_Γ be its flow semigroup. Then $\#_G(S_\Gamma) = n - 2$.

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2-vertex connected $\Longleftrightarrow$ does not disconnect by removing a vertex.

Theorem (Horváth, Nehaniv, Podoski)

Let Γ be a 2-vertex connected graph on n points, and let S_Γ be its flow semigroup. Then $\#_G(S_\Gamma) = n - 2$.

Open problem

What is the complexity for 2-edge connected but not 2-vertex connected graphs?

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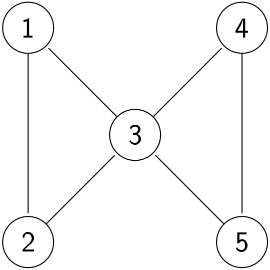
Example (Bowtie graph, $2 \leq \#_G(S_\Gamma) \leq 3$)



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