On large rigid sets of monounary algebras

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Motivation

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- For A, $B \in C$ put
$$A \succeq_{\mathfrak{C}} B$$
if there exists $h \in H$ such that h is a morphism of A into B.
- $\succeq_{\mathfrak{C}}$ is a quasiorder on C.

- $A, B \in C$ are said to be *incomparable*, if neither $A \succeq_{\mathfrak{C}} B$ nor $B \succeq_{\mathfrak{C}} A$.
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- **How large antichains?**

Isomorphisms, epimorphisms

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$\mathfrak{C} = (C, H)$

- $C=$ class of all algebraic structures of given type,
  $H=$ all isomorphisms (epimorphisms, respectively) of A onto B, for $A, B \in C$.

Clearly, ARBITRARILY large antichains.

Isomorphisms, given cardinality

- $C=$ class of all algebraic structures of given type τ and given infinite cardinality $\mathfrak{m}$

Assume that k_i, $i \in \mathbb{N}_0$ is the number of all operation symbols of arity i, n_i, $i \in \mathbb{N}$ the number of all relation symbols of arity i (in τ).

Theorem (Comer, LeTourneau)

There exist

$$2^{\sum_{i\in\mathbb{N}}(k_i+n_i)\mathfrak{m}^i} \cdot \mathfrak{m}^{k_0}$$

non-isomorphic $\mathfrak{m}$-element algebras of type τ.
Specially, there exist $2^{\mathfrak{m}}$ non-isomorphic and rigid $\mathfrak{m}$-element monounary algebras.

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- *rigid* algebra: no automorphisms except identity

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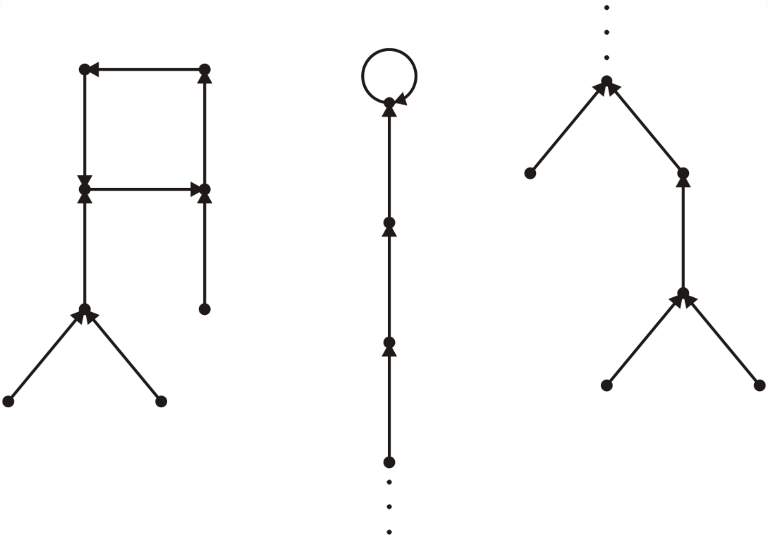
- $c \in A$ is a *cyclic* element of (A, f) if $f^k(c) = c$ for some $k \in \mathbb{N}$
- the set of all cyclic elements of some connected component of (A, f) is a *cycle* of (A, f)



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There exists an antichain M in the system of all connected monounary algebras such that card $M = \mathfrak{c}$*. (Moreover, each algebra of M is countable.)*

Hence, $\mathfrak{c}$ is the best possible upper bound for the number of pairwise incomporable connected monounary algebras.

Embeddings

C = class of all algebraic structures of given type τ
$M \subseteq C$ is said to be *rigid with respect to embeddability* (*e-rigid*, for short), if whenever $A, B \in M$, $\varphi : A \to B$ is an embedding (that is, injective homomorphism), then

- $A = B$ and
- φ is the identity map id_A.

Note: A is an *e-rigid algebra* if it is an algebra and $\{A\}$ is an e-rigid set.

Auxiliaries

An infinite cardinal $\mathfrak{m}$ is called *inaccessible* if the following three conditions hold:

- $\aleph_0 < \mathfrak{m}$;
- for all cardinals $\mathfrak{n}$, if $\mathfrak{n} < \mathfrak{m}$, then $2^{\mathfrak{n}} < \mathfrak{m}$;
- $\mathfrak{m}$ is a *regular cardinal*, that is, for every set I of cardinals, if $|I| < \mathfrak{m}$ and all members of I are smaller than $\mathfrak{m}$, then $\sum_{\mathfrak{n} \in I} \mathfrak{n} < \mathfrak{m}$.

Large rigid sets of monounary algebras

Note: there exists a model of set theory in which there exists no inaccessible cardinal. In this model, our theorems hold for all cardinals $\mathfrak{m}$.

Theorem

Let $\mathfrak{m}$ be a cardinal such that there is no inaccessible cardinal $\mathfrak{k}$ with $\mathfrak{k} \leq \mathfrak{m}$. Then there exists an e-rigid set M of monounary algebras such that $|M| = \mathfrak{m}$.

General case

Theorem

Let τ be a similarity type of algebras containing an at least unary operation, and let $\mathfrak{m}$ be a cardinal number. If there is no inaccessible cardinal $\mathfrak{k}$ such that $\mathfrak{k} \leq \mathfrak{m}$, then there exists an e-rigid set M of τ-algebras such that $|M| = \mathfrak{m}$.

Thank you for your attention!

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https://en.wikipedia.org/wiki/Consolida regalis
(Forking Larkspur)