Monads and algebras

Gejza Jenča

September 9, 2016

The Plan

- Example: the free monoid monad.

The Plan

- Example: the free monoid monad.
- Algebras are algebras.

The Plan

- Example: the free monoid monad.
- Algebras are algebras.
- Algebras in graphs.

The Plan

- Example: the free monoid monad.
- Algebras are algebras.
- Algebras in graphs.
- Algebras in maps.

The Plan

- Example: the free monoid monad.
- Algebras are algebras.
- Algebras in graphs.
- Algebras in maps.
- Comonads and coalgebras.

Disclaimers

- I will try not to tell you anything about

Disclaimers

- I will try not to tell you anything about
  - adjunctions,

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.
- It can be considered as an attempt to reformulate universal algebra in categorical terms.

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.
- It can be considered as an attempt to reformulate universal algebra in categorical terms.
- Almost everything I will tell you is folklore

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.
- It can be considered as an attempt to reformulate universal algebra in categorical terms.
- Almost everything I will tell you is folklore (in some circles);

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.
- It can be considered as an attempt to reformulate universal algebra in categorical terms.
- Almost everything I will tell you is folklore (in some circles); sometimes the circles are very disjoint.

Disclaimers

- I will try not to tell you anything about
  - adjunctions,
  - Lawvere theories.
- The theory of monads was created in the 1960's by Mac Lane, Eilenberg, Moore, Beck, Freyd, Barr and others.
- It can be considered as an attempt to reformulate universal algebra in categorical terms.
- Almost everything I will tell you is folklore (in some circles); sometimes the circles are very disjoint.
- Except for the part about graphs; that one appears to be new.

What is a monad?

- A monad is a quadruple of things consisting of

What is a monad?

- A monad is a quadruple of things consisting of
  - a category,

What is a monad?

- A monad is a quadruple of things consisting of
  - a category,
  - an endofunctor,

What is a monad?

- A monad is a quadruple of things consisting of
  - a category,
  - an endofunctor,
  - a unit,

What is a monad?

- A monad is a quadruple of things consisting of
  - a category,
  - an endofunctor,
  - a unit,
  - a multiplication.

The category

The category

Set

The endofunctor

- Let X be a set.

The endofunctor

- Let X be a set.
- Let X^* be the set of all words over X – (the underlying set of) the free monoid over X.

The endofunctor

- Let X be a set.
- Let X^* be the set of all words over X – (the underlying set of) the free monoid over X.
- Note that the rule $X \mapsto X^*$ gives us a functor **Set** $\to$ **Set** that acts on mappings as follows:

The endofunctor

- Let X be a set.
- Let X^* be the set of all words over X – (the underlying set of) the free monoid over X.
- Note that the rule $X \mapsto X^*$ gives us a functor **Set** $\to$ **Set** that acts on mappings as follows:
- for $f : X \to Y$, the mapping $f^* : X^* \to Y^*$ is given by the rule

$$f^*([a_1 a_2 \dots a_n]) = [f(a_1) \dots f(a_n)]$$

The unit

- What is a natural map $\eta_X : X \to X^*$?

The unit

- What is a natural map $\eta_X : X \to X^*$?
- Take $a \in X$ and map it to the one-letter word $[a]$:

$$\eta_X(a) = [a]$$

The unit

- What is a natural map $\eta_X : X \to X^*$?
- Take $a \in X$ and map it to the one-letter word $[a]$:

$$\eta_X(a) = [a]$$

- We claim that, for every mapping $f : X \to Y$ of sets, the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ Y & \xrightarrow[\eta_Y]{} & Y^* \end{array}$$

commutes.

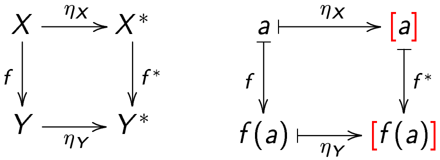
The unit

- Why?

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X^* \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\
Y & \xrightarrow[\eta_Y]{} & Y^*
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xmapsto{\eta_X} & [a] \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\
f(a) & \xmapsto[\eta_Y]{} & [f(a)]
\end{array}
$$

The unit

- Why?



- So η is a natural transformation $\mathrm{id}_{\mathbf{Set}} \to (\,)^*$.

The multiplication

- What is a natural map $\mu_X : (X^*)^* \to X^*$?

The multiplication

- What is a natural map $\mu_X : (X^*)^* \to X^*$?
- The elements of $(X^*)^*$ are "words of words".
- So it has elements like

$$[[aba][cabcc][bbc]]$$

The multiplication

- What is a natural map $\mu_X : (X^*)^* \to X^*$?
- The elements of $(X^*)^*$ are "words of words".
- So it has elements like

$$[[aba][cabcc][bbc]]$$

- We can just concatenate the inner words:

$$\mu_X([[aba][cabcc][bbc]]) = [abacabccbbc]$$

The multiplication

- What is a natural map $\mu_X : (X^*)^* \to X^*$?
- The elements of $(X^*)^*$ are "words of words".
- So it has elements like

$$[[aba][cabcc][bbc]]$$

- We can just concatenate the inner words:

$$\mu_X([[aba][cabcc][bbc]]) = [abacabccbbc]$$

- Note that μ_X behaves like a "flattening".

The multiplication

- Is μ natural?
- Yes:

$$\begin{array}{ccc} (X^*)^* & \xrightarrow{\mu_X} & X^* \\ \Big\downarrow{\scriptstyle (f^*)^*} & & \Big\downarrow{\scriptstyle f^*} \\ (Y^*)^* & \xrightarrow[\mu_Y]{} & Y^* \end{array}$$

The multiplication

- Is μ natural?
- Yes:

$$\begin{array}{ccc} (X^*)^* & \xrightarrow{\mu_X} & X^* \\ \Big\downarrow (f^*)^* & & f^* \Big\downarrow \\ (Y^*)^* & \xrightarrow[\mu_Y]{} & Y^* \end{array}$$

- For example,

$$\begin{array}{ccc} [[ab][ac]] & \longmapsto & [abac] \\ \Big\downarrow & & \Big\downarrow \\ [[f(a)f(b)][f(a)f(c)]] & \longmapsto & [f(a)f(b)f(a)f(c)] \end{array}$$

$(\mathcal{C}, \eta, \mu)$

. So we have data of the following type:

$(\mathcal{C}, \eta, \mu)$

. So we have data of the following type:

- a category $\mathcal{C}$,

$(\mathcal{C}, \eta, \mu)$

. So we have data of the following type:

- a category $\mathcal{C}$,
- a functor $T : \mathcal{C} \to \mathcal{C}$,

$(\mathcal{C}, \eta, \mu)$

. So we have data of the following type:

- a category $\mathcal{C}$,
- a functor $T : \mathcal{C} \to \mathcal{C}$,
- a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to T$,

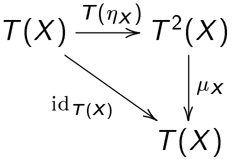
$(\mathcal{C}, \eta, \mu)$

. So we have data of the following type:

- a category $\mathcal{C}$,
- a functor $T : \mathcal{C} \to \mathcal{C}$,
- a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to T$,
- a natural transformation $\mu : T^2 \to T$.

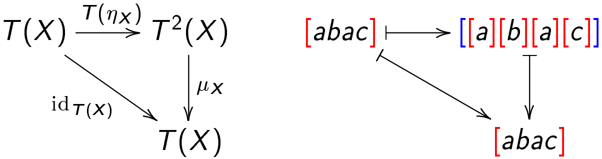
Axioms of a monad

Right unit axiom



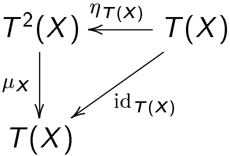
Axioms of a monad

Right unit axiom



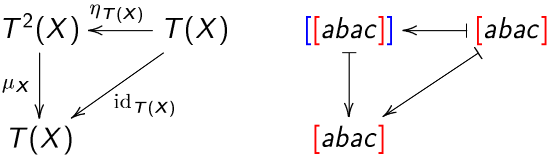
Axioms

Left unit axiom



Axioms

Left unit axiom



Axioms of a monad

Associativity axiom

$$
\begin{array}{ccc}
T^3(X) & \xrightarrow{T(\mu_X)} & T^2(X) \\
\Big\downarrow{\scriptstyle \mu_{T(X)}} & & \Big\downarrow{\scriptstyle \mu_X} \\
T^2(X) & \xrightarrow{\mu_X} & T(X)
\end{array}
$$

Axioms of a monad

Associativity axiom

$$\begin{array}{ccc} T^3(X) & \xrightarrow{T(\mu_X)} & T^2(X) \\ \downarrow{\scriptstyle \mu_{T(X)}} & & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow{\mu_X} & T(X) \end{array}$$

$$\begin{array}{ccc} [[[ab][bc]]\,[[ca]]] & \longmapsto & [[abbc][ca]] \\ \downarrow & & \downarrow \\ [[ab][bc][ca]] & \longmapsto & [abbcca] \end{array}$$

Usual definition

Definition

Let $\mathcal{C}$ be a category. A monad over $\mathcal{C}$ is a triple (T, η, μ) such that

- $T : \mathcal{C} \to \mathcal{C}$,
- $\eta : \mathrm{id}_{\mathcal{C}} \to T$,
- $\mu : T^2 \to T$
- such that the unit and associativity axioms hold.

Slick definition

Definition

A monad is a monoid in the monoidal category of endofunctors of a category.

The free monoid example works in a much more general setting:

- Take a variety $\mathcal{V}$.

The free monoid example works in a much more general setting:

- Take a variety $\mathcal{V}$.
- Write $F_{\mathcal{V}}$ for the endofunctor of **Set** that takes a set X to the (underlying set of) $\mathcal{V}$-free algebra over X.

The free monoid example works in a much more general setting:

- Take a variety $\mathcal{V}$.
- Write $F_{\mathcal{V}}$ for the endofunctor of **Set** that takes a set X to the (underlying set of) $\mathcal{V}$-free algebra over X.
- Write η for the obvious natural transformation that "embeds variables".

The free monoid example works in a much more general setting:

- Take a variety $\mathcal{V}$.
- Write $F_{\mathcal{V}}$ for the endofunctor of **Set** that takes a set X to the (underlying set of) $\mathcal{V}$-free algebra over X.
- Write η for the obvious natural transformation that “embeds variables“.
- Write μ for the obvious natural transformation that “flattens terms over terms“.

The free monoid example works in a much more general setting:

- Take a variety $\mathcal{V}$.
- Write $F_{\mathcal{V}}$ for the endofunctor of **Set** that takes a set X to the (underlying set of) $\mathcal{V}$-free algebra over X.
- Write η for the obvious natural transformation that "embeds variables".
- Write μ for the obvious natural transformation that "flattens terms over terms".

Then $(F_{\mathcal{V}}, \eta, \mu)$ is a monad over **Set**.

Algebras for a monad

Definition

Let (T, η, μ) be a monad over $\mathcal{C}$. Then an algebra for T is a pair (X, α), where

- X is an object of $\mathcal{C}$ and

Algebras for a monad

Definition

Let (T, η, μ) be a monad over $\mathcal{C}$. Then an algebra for T is a pair (X, α), where

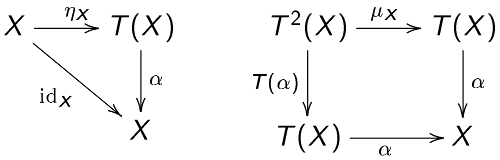
- X is an object of $\mathcal{C}$ and
- $\alpha : T(X) \to X$

Algebras for a monad

Definition

Let (T, η, μ) be a monad over $\mathcal{C}$. Then <u>an algebra for T</u> is a pair (X, α), where

- X is an object of $\mathcal{C}$ and
- $\alpha : T(X) \to X$
- such that the following diagrams commute

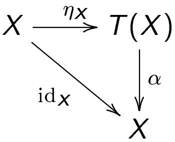


What are algebras for the free monoid monad?

- An algebra over a set X equips X a mapping $\alpha : X^* \to X$.

What are algebras for the free monoid monad?

- An algebra over a set X equips X a mapping $\alpha : X^* \to X$.
- The commutative triangle



just says that

$$\alpha([a]) = a$$

What does the square

$$\begin{array}{ccc} T^2(X) & \xrightarrow{\mu_X} & T(X) \\ {\scriptstyle T(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ T(X) & \xrightarrow[\alpha]{} & X \end{array}$$

tell us about α?
The square is a machine for proving equalities!

Let us define a binary operation $\odot : X \times X \to X$ by the rule $p \odot q := \alpha([pq])$.

Let us define a binary operation $\odot : X \times X \to X$ by the rule $p \odot q := \alpha([pq])$.
We claim that $\odot$ is associative.

Let us define a binary operation $\odot : X \times X \to X$ by the rule $p \odot q := \alpha([pq])$.
We claim that $\odot$ is associative.

Proof.
Let us plug the term $[[ab][c]]$ to the top-left corner of the square.

Let us define a binary operation $\odot : X \times X \to X$ by the rule
$p \odot q := \alpha([pq])$.
We claim that $\odot$ is associative.

Proof.

Let us plug the term $[[ab][c]]$ to the top-left corner of the square.

$$\begin{array}{ccc}
[[ab][c]] & \xmapsto{\mu_X} & [abc] \\
{\scriptstyle T(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\
[\alpha([ab])\alpha([c])] = [(a \odot b)\, c] & \xmapsto[\alpha]{} & (a \odot b) \odot c = \alpha([abc])
\end{array}$$

This proves the equality in the bottom-right corner of the diagram. $\square$

Analogously, we may plug $[[a][bc]]$ to the top left corner of the diagram to prove that $a \odot (b \odot c) = \alpha([abc])$, so the associativity axiom holds.

Let us put $e := \alpha([])$.
Chasing $[[a][]]$ around the square

$$
\begin{array}{ccc}
[[a][]] & \longmapsto & [a] \\
\big\downarrow & & \big\downarrow \\
[\alpha([a])\alpha([])] = [a, e] & \longmapsto & \alpha([a, e]) = \alpha([a])
\end{array}
$$

gives us the equality $a \odot e = a$. Similarly, we get $e \odot a = a$.

Algebras are algebras

- Thus, every algebra (X, α) for the free monoid monad gives rise to a monoid $(X, \odot, e)$.

Algebras are algebras

- Thus, every algebra (X, α) for the free monoid monad gives rise to a monoid $(X, \odot, e)$.
- On the other hand, if we start with a monoid $(X, \odot, e)$ and define $\alpha : X^* \to X$ by the rules $\alpha([]) = e$ and

$$\alpha([a_1 \dots a_n]) = a_1 \odot \cdots \odot a_n$$

then (X, α) is an algebra for the free monoid monad.

So, we may identify algebras for the free monoid monad with monoids.
Again, this works for every variety of algebras.

Morphisms of algebras

Morphisms of algebras

Definition

If (A, α), (B, β) are algebras for a monad T, then a morphism of algebras $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in the underlying category such that the square

$$
\begin{array}{ccc}
T(A) & \xrightarrow{T(f)} & T(B) \\
\alpha \big\downarrow & & \big\downarrow \beta \\
A & \xrightarrow[f]{} & B
\end{array}
$$

commutes.

This definition works as intended; we obtain exactly the usual notion of morphism of algebras.

What is it good for?

- The definitions of monads and their algebras are given entirely in categorical terms.

What is it good for?

- The definitions of monads and their algebras are given entirely in categorical terms.
- So, we have generalized the notion of an algebra in a (particular) variety from "a set such that ..."

What is it good for?

- The definitions of monads and their algebras are given entirely in categorical terms.
- So, we have generalized the notion of an algebra in a (particular) variety from “a set such that ...”to “an object in a category such that ...”.

What is it good for?

- The definitions of monads and their algebras are given entirely in categorical terms.
- So, we have generalized the notion of an algebra in a (particular) variety from “a set such that ...”to “an object in a category such that ...”.
- This gives us a proper framework to speak about things like “topological/partially ordered/whatever monoids/groups/whatever”.

What is it good for?

- The definitions of monads and their algebras are given entirely in categorical terms.
- So, we have generalized the notion of an algebra in a (particular) variety from "a set such that ..."to "an object in a category such that ...".
- This gives us a proper framework to speak about things like "topological/partially ordered/whatever monoids/groups/whatever".
- But sometimes, the algebras for a monad do not look like algebras, at all.

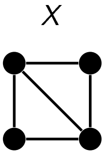
The category of graphs

Definition

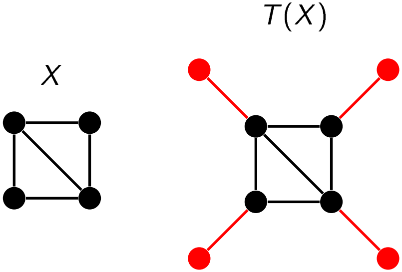
A graph G is a set $V(G)$ of vertices, equipped with a system of two-element sets $E(G)$, called edges.

A morphism of graphs $f : A \to B$ is a set-mapping $f : V(A) \to V(B)$ such that $\{x, y\} \in E(A)$ implies $\{f(x), f(y)\} \in E(B)$.

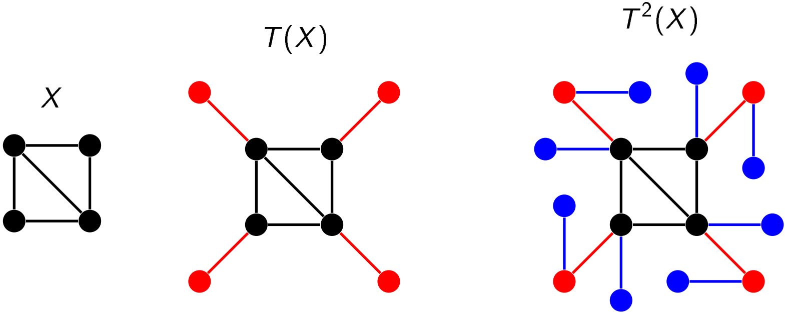
A monad on graphs



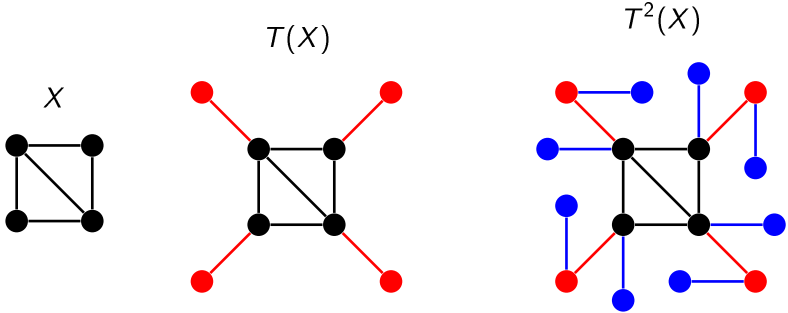
A monad on graphs



A monad on graphs



A monad on graphs



$$X \xrightarrow{\eta_X} T(X) \xleftarrow{\mu_X} T^2(X)$$

What are the algebras?

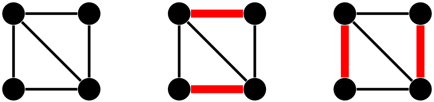
$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & T(X) \\
 & \searrow^{\mathrm{id}_X} & \downarrow \alpha \\
 & & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
T^2(X) & \xrightarrow{\mu_X} & T(X) \\
T(\alpha) \downarrow & & \downarrow \alpha \\
T(X) & \xrightarrow[\alpha]{} & X
\end{array}
$$

Perfect matchings

Definition

A perfect matching on a graph A is a subset M of the set of edges of A such that every vertex is in exactly one edge from M.

Example



Similar monads

There are similar monads on the category of graphs, their algebras are:

Similar monads

There are similar monads on the category of graphs, their algebras are:

- (edge) packing of triangles,

Similar monads

There are similar monads on the category of graphs, their algebras are:

- (edge) packing of triangles,
- (vertex) disjoint cycle cover.

Similar monads

There are similar monads on the category of graphs, their algebras are:

- (edge) packing of triangles,
- (vertex) disjoint cycle cover.

Both of these are well-known things.

The arrow category $\mathbf{Set}^{\rightarrow}$

- Objects: mappings in **Set**.
- Morphisms: commutative squares.

The arrow category $\mathbf{Set}^{\rightarrow}$

- Objects: mappings in **Set**.
- Morphisms: commutative squares.

Explicitly,

- for $f : A_1 \rightarrow A_2$

The arrow category $\mathbf{Set}^{\rightarrow}$

- Objects: mappings in **Set**.
- Morphisms: commutative squares.

Explicitly,

- for $f : A_1 \rightarrow A_2$
- $g : B_1 \rightarrow B_2$

The arrow category $\mathbf{Set}^{\rightarrow}$

- Objects: mappings in **Set**.
- Morphisms: commutative squares.

Explicitly,

- for $f : A_1 \to A_2$
- $g : B_1 \to B_2$
- a morphism $f \to g$ is a pair of mappings

The arrow category $\mathbf{Set}^{\rightarrow}$

- Objects: mappings in **Set**.
- Morphisms: commutative squares.

Explicitly,

- for $f : A_1 \to A_2$
- $g : B_1 \to B_2$
- a morphism $f \to g$ is a pair of mappings
- $h_1 : A_1 \to B_1$ and $h_2 : A_2 \to B_2$ such that

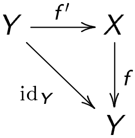
$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ B_1 & \xrightarrow[g]{} & B_2 \end{array}$$

commutes.

Retractions

Definition

Let $f : X \to Y$ be a mapping in **Set**. A mapping $f' : Y \to X$ is a <u>retraction of f</u> if the diagram



commutes.

Free retractions

- Let $f : X \to Y$ be an object of $\mathbf{Set}^{\to}$.
- What could a "free retraction" over f be?

The free retraction monad

The endofunctor

The endofunctor $T : \mathbf{Set}^{\rightarrow} \rightarrow \mathbf{Set}^{\rightarrow}$ takes every object $f : X \rightarrow Y$ of $\mathbf{Set}^{\rightarrow}$ to its "extension by the id_Y"

$$T(\ X \xrightarrow{f} Y\ ) = (\ Y \oplus X \xrightarrow{\langle \mathrm{id}_Y, f \rangle} Y\ )$$

and acts on morphisms of $\mathbf{Set}^{\rightarrow}$ as follows:

$$\begin{array}{ccc}
A_1 & \xrightarrow{f} & A_2 \\
{\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\
B_1 & \xrightarrow[g]{} & B_2
\end{array}
\quad \overset{T}{\longmapsto} \quad
\begin{array}{ccc}
A_2 \oplus A_1 & \xrightarrow{\langle \mathrm{id}_{A_2}, f \rangle} & A_2 \\
{\scriptstyle h_2 \oplus h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\
B_2 \oplus B_1 & \xrightarrow[\langle \mathrm{id}_{B_2}, g \rangle]{} & B_2
\end{array}$$

The free retraction monad

The unit and the multiplication

η_f

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
\Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle \mathrm{id}_Y} \\
Y \oplus X & \xrightarrow{\langle \mathrm{id}_Y , f \rangle} & Y
\end{array}
$$

μ_f

$$
\begin{array}{ccc}
Y \oplus Y \oplus X & \xrightarrow{\langle \mathrm{id}_Y , \mathrm{id}_Y , f \rangle} & Y \\
\Big\downarrow{\scriptstyle \langle \nabla_Y , \mathrm{id}_X \rangle} & & \Big\downarrow{\scriptstyle \mathrm{id}_Y} \\
Y \oplus X & \xrightarrow[\mathrm{id}_Y \oplus f]{} & X
\end{array}
$$

The free retraction monad

Algebras

An algebra for the free retraction monad is a commutative square

$$\begin{array}{ccc} Y \oplus X & \xrightarrow{\langle \mathrm{id}_Y, f \rangle} & Y \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The free retraction monad

Algebras

An algebra for the free retraction monad is a commutative square

$$\begin{array}{ccc} Y \oplus X & \xrightarrow{\langle \mathrm{id}_Y, f \rangle} & Y \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Moreover, the properties from the definition of an algebra for a monad imply that

- $\alpha_2 = \mathrm{id}_Y$
- α_1 is equal to id_X on X

This implies that the algebras over $f : X \to Y$ are in a one-to-one correspondence with certain mappings $f' : Y \to X$.

The free retraction monad

Algebras

$$
\begin{array}{ccc}
Y \oplus X & \xrightarrow{\mathrm{id}_Y \oplus f} & Y \\
{\scriptstyle \langle f', \mathrm{id}_X \rangle}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

It turns out that the square is an algebra for the free retraction monad iff $f \circ f' = \mathrm{id}_Y$.

Comonads

Definition

A monad on the category $\mathcal{C}^{op}$ is called a comonad on the category $\mathcal{C}$.

Comonads

Unwinding this definition, this means that a comonad on $\mathcal{C}$ consists of a triple (S, ϵ, σ) such that

- S is an endofunctor,
- $\epsilon : S \to \mathrm{id}_{\mathcal{C}}$,
- $\sigma : S \to S^2$

satisfying the conditions dual to the conditions in the definition of a monad.

A comonad on **Set**

- Consider the endofunctor "free semigroup", that takes a set X to the set of all nonempty words over X, denoted by X^+.
- What is a natural mapping $\epsilon_X : X^+ \to X$?
- What is a natural mapping $\sigma_X : X^+ \to (X^+)^+$?

A comonad on **Set**

- Consider the endofunctor "free semigroup", that takes a set X to the set of all nonempty words over X, denoted by X^+.
- What is a natural mapping $\epsilon_X : X^+ \to X$?
- What is a natural mapping $\sigma_X : X^+ \to (X^+)^+$?

Possible answers:

$$\epsilon_X([x_1 \ldots x_n]) = x_1$$

A comonad on **Set**

- Consider the endofunctor "free semigroup", that takes a set X to the set of all nonempty words over X, denoted by X^+.
- What is a natural mapping $\epsilon_X : X^+ \to X$?
- What is a natural mapping $\sigma_X : X^+ \to (X^+)^+$?

Possible answers:

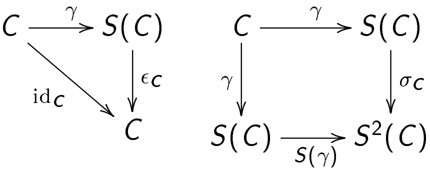
$$\epsilon_X([x_1 \dots x_n]) = x_1$$

$$\sigma_X([x_1 \dots x_n]) = [[x_1 \dots x_n][x_2 \dots x_n] \dots [x_n]]$$

Coalgebras

Definition

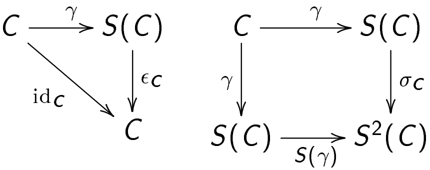
A coalgebra for a comonad (S, ϵ, σ) is a pair (C, γ), where $\gamma : C \to S(C)$ satisfying the diagrams dual to the diagrams in the definition of a monad:



Coalgebras

Definition

A coalgebra for a comonad (S, ϵ, σ) is a pair (C, γ), where $\gamma : C \to S(C)$ satisfying the diagrams dual to the diagrams in the definition of a monad:



The coalgebras for $()^+$ are "directed forests with finite branches"; γ takes a vertex to its branch.

There are other monads with the same endofunctor and ϵ, for example:

There are other monads with the same endofunctor and ϵ, for example:

- $\sigma_X([x_1 x_2 \dots x_n]) = [[x_1 x_2 \dots x_n][x_2 x_3 \dots x_n] \dots [x_n x_1 \dots x_2]]$, the algebras are disjoint unions of directed cycles.

There are other monads with the same endofunctor and ϵ, for example:

- $\sigma_X([x_1 x_2 \dots x_n]) = [[x_1 x_2 \dots x_n][x_2 x_3 \dots x_n] \dots [x_n x_1 \dots x_2]]$, the algebras are disjoint unions of directed cycles.
- $\sigma_X([x_1 x_2 \dots x_n]) = [[x_1 x_2 \dots x_n][x_2][x_3] \dots [x_n]]$, the algebras are "partitions with a fixed nonempty subset in each of the blocks".

Thank you for your attention.