Selected extensions of basic algebras

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The concept of basic algebra was introduced as a common generalization of an MV-algebra and an orthomodular lattice. Remember that MV-algebras serve as an algebraic axiomatization of the so-called Łukasiewicz many-valued logics and orthomodular lattices form an algebraic counterpart of the logic of quantum mechanics. Hence, basic algebras form a common algebraic axiomatization of both logics mentioned above.

Recall that a **basic algebra** is an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(B1) $x \oplus 0 = x$

(B2) $\neg\neg x = x$

(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, where $1 = \neg 0$.

Every basic algebra has its second face, namely $\mathscr{A} = (A; \oplus, \neg, 0)$ can be organized into a bounded lattice $(A; \vee, \wedge)$, where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y),$$

whose order is given by

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Of course, $0 \leq x \leq 1$ for each $x \in A$. Moreover, this lattice $(A; \vee, \wedge)$ is endowed by a set $(^a)_{a \in A}$ of so-called **sectional antitone involutions**, i.e. for each $a \in A$ there exists a mapping $x \mapsto x^a$ of the interval $[a, 1]$ (called **section**) into itself such that

$$x^{aa} = x \quad \text{and} \quad x \leq y \Rightarrow y^a \leq x^a \quad \text{for all } x, y \in [a, 1].$$

This system $\mathscr{L}(\mathscr{A}) = (L; \vee, \wedge, (^a)_{a \in L}, 0, 1)$ is called a **lattice with sectional antitone involutions** assigned to $\mathscr{A} = (A; \oplus, \neg, 0)$. Also conversely, having a bounded lattice with sectional antitone involutions $\mathscr{L} = (L; \vee, \wedge, (^a)_{a \in L}, 0, 1)$, one can convert it into a basic algebra $\mathscr{A}(\mathscr{L}) = (L; \oplus, \neg, 0)$, where

$$\neg x = x^0 \quad \text{and} \quad x \oplus y = (\neg x \vee y)^y.$$

Moreover, the assignments $\mathscr{A} \to \mathscr{L}(\mathscr{A})$ and $\mathscr{L} \to \mathscr{A}(\mathscr{L})$ are one-to-one correspondences, i.e. $\mathscr{A}(\mathscr{L}(\mathscr{A})) = \mathscr{A}$ and $\mathscr{L}(\mathscr{A}(\mathscr{L})) = \mathscr{L}$.

Outline

Abstract – Pseudo basic algebras

Since basic algebras are equivalent to bounded lattices with sectional antitone involutions, it motivated us to study an algebraic counterpart of semilattices with sectional switching involutions. These algebras are called pseudo basic algebras. They are determined by four independent identities. Several basic properties of these algebras are presented and a particular interest is devoted to pseudo basic algebras whose main involution is even antitone (so-called strict pseudo basic algebras) and to those whose binary operation is commutative.

Basic concepts

Consider a section $[a, 1]$ of an ordered set with greatest element 1. A mapping $x \mapsto x^a$ of $[a, 1]$ into itself is called a **sectional switching involution** if $x^{aa} = x$ for each $x \in [a, 1]$ and $a^a = 1, 1^a = a$. In general, we do not ask that this involution should be antitone; it only switches the endpoints of the section.

Now, we can consider a bounded lattice with sectional switching involutions $\mathscr{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ and study what an algebra can be obtained by using a similar construction as that for basic algebras. For our purposes, we will consider only a semilattice since the operation meet is not applied in the construction of the operations of the new algebra.

Theorem 1

Let $\mathscr{S} = (S; \vee, ({}^a)_{a\in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions. Define $\neg x = x^0$ and $x \oplus y = (x^0 \vee y)^y$. Then the algebra $\mathscr{A}(\mathscr{S}) = (S; \oplus, \neg, 0)$ assigned to $\mathscr{S}$ satisfies the following identities:

(P1) $\neg x \oplus x = 1$

(P2) $x \oplus 0 = x$

(P3) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$

(P4) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x$.

If, moreover, the involution $x \mapsto x^0$ is antitone, then $\mathscr{A}(\mathscr{S})$ satisfies the identity

(A) $(\neg(x \oplus y) \oplus y) \oplus x = 1$.

Note that the axioms (P1)–(P4) are independent.

Remark 1 (1/2)

(a) Every bounded semilattice $\mathscr{S} = (S; \vee, 0, 1)$ can be considered as a semilattice with sectional switching involutions. Namely, for each $a \in S$ one can define a switching involution on $[a, 1]$ as follows: $a^a = 1, 1^a = a$ and $x^a = x$ for each $x \in [a, 1], a \neq x \neq 1$.

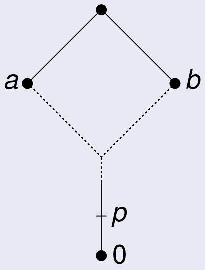
(b) If the involution $x \mapsto x^0$ is antitone, then $\mathscr{S} = (S; \vee, 0, 1)$ is in fact a lattice due to the DeMorgan laws because

$$x \wedge y = (x^0 \vee y^0)^0.$$

(c) There exist bounded semilattices with sectional switching involutions which are not lattices.

Remark 1 (2/2)

Such a semilattice $\mathscr{K}$, call a “kite” is visualized in Figure 1:



It is an ordinal sum of an infinite chain C with the least element 0 and without a greatest element and a three element semilattice $\{a, b, 1\}$, i.e. $p \leq a, b$ for each $p \in C$. Then $\mathscr{K}$ is a $\vee$-semilattice which is not a lattice since $\inf\{a, b\}$ does not exists. Moreover, for each $c \in \mathscr{K}$ we define $c^c = 1, 1^c = c$ and $x^c = x$ for $x \in [c, 1]$, $c \neq x \neq 1$.

We have shown that to every bounded semilattice $\mathscr{S}$ with sectional switching involutions we can assign an algebra $\mathscr{A}(\mathscr{S}) = (S; \oplus, \neg, 0)$ satisfying (P1)–(P4). We are going to show that algebras satisfying (P1)–(P4) are interesting for their own sake.

Definition 1

An algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities (P1)–(P4) will be called a **pseudo basic algebra**. If, moreover, $\mathscr{A}$ satisfies also the identity (A), it will be called a **strict pseudo basic algebra**.

Theorem 2

The variety of pseudo basic algebras is weakly regular. The variety of strict pseudo basic algebras is congruence regular and arithmetical.

Our next task is to show that also conversely, every pseudo basic algebra can be organized into a bounded semilattice with sectional switching involutions.

Theorem 3

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra. Define $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and for any $a \in A$, let $x^a = \neg x \oplus a$. Then

(a) $(A; \vee)$ is a join-semilattice with least element 0 and greatest element 1

(b) $x \leq y$ if and only if $\neg x \oplus y = 1$ is the induced order of the semilattice $(A; \vee)$

(c) for each $a \in A$ and $x \in [a, 1]$, the mapping $x \mapsto x^a = \neg x \oplus a$ is a sectional switching involution on the section $[a, 1]$.

If, moreover, $\mathscr{A}$ is a strict pseudo basic algebra, then $(A; \vee)$ is a lattice where $x \wedge y = \neg(\neg x \vee \neg y)$.

We can show that the assignment between pseudo basic algebras and bounded semilattices with sectional switching involutions is a one-to-one correspondence.

Theorem 4

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, $\mathscr{S}(\mathscr{A})$ its assigned semilattice with sectional switching involutions. Then $\mathscr{A}(\mathscr{S}(\mathscr{A})) = \mathscr{A}$.
Let $\mathscr{S} = (S; \vee, (^a)_{a \in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions, $\mathscr{A}(\mathscr{S})$ its assigned pseudo basic algebra. Then $\mathscr{S}(\mathscr{A}(\mathscr{S})) = \mathscr{S}$.

Outline

Now, we reveal several interesting properties of strict and/or commutative pseudo basic algebras.

As mentioned in (b) of Remark 1, if a pseudo basic algebra $\mathscr{A}$ is strict, then the assigned semilattice $\mathscr{S}(\mathscr{A})$ is a lattice $(A; \vee, \wedge)$ where $x \wedge y = \neg(\neg x \vee \neg y)$. In what follows, we will use this fact and $\mathscr{S}(\mathscr{A})$ will be called an **assigned lattice**.

Theorem 5

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a strict pseudo basic algebra. Then $\neg$ is a complementation in the induced lattice $(A; \vee, \wedge, 0, 1)$ if and only if $\mathscr{A}$ satisfies the identity $x \oplus x = x$.

A pseudo basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is called **commutative** if it satisfies the identity $x \oplus y = y \oplus x$ and $\mathscr{A}$ is called **associative** if it satisfies the identity $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
An interesting connection is given by the following.

Theorem 6

(a) Every commutative pseudo basic algebra is strict.
(b) A pseudo basic algebra is an MV-algebra if and only if it is associative.

It was proved by M. Botur and R. Halaš that every finite commutative basic algebra is in fact an MV-algebra. Hence, it is a natural question if there really exist commutative pseudo basic algebras which are not basic algebras. The answer is positive also for a finite pseudo basic algebra.

Example 1 (1/2)

Consider the commutative pseudo basic algebra $\mathscr{A} = (\{0, a, b, c, d, e, 1\}; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables

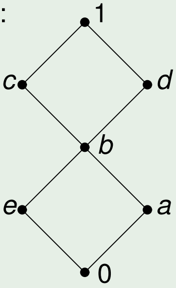
x	0	a	b	c	d	e	1
$\neg x$	1	c	b	a	e	d	0

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	c	d	1	d	b	1
b	b	d	1	1	1	c	1
c	c	1	1	1	1	c	1
d	d	d	1	1	1	1	1
e	e	b	c	c	1	d	1
1	1	1	1	1	1	1	1

By Theorem 6 (a), $\mathscr{A}$ is strict and its induced lattice is visualized in Figure 2.

Example 1 (2/2)

Figure 2:



The sectional switching involutions are as follows:
In $[0,1]$, $x^0 = \neg x$.
In $[e,1]$ it is: $e^e = 1$, $b^e = c$, $c^e = b$, $d^e = d$, $1^e = e$.
In $[a,1]$ it is: $a^a = 1$, $b^a = d$, $d^a = b$, $c^a = c$, $1^a = a$.
In $[b,1]$ it is: $b^b = 1$, $c^b = d$, $d^b = c$ and $1^b = b$.
In $[c,1],[d,1]$ and $[1,1]$ it is determined uniquely. One can easily check that $x \mapsto x^e$ and $x \mapsto x^a$ are not antitone since e.g.

$$b \leq d \quad \text{but} \quad b^e = c \parallel d = d^e.$$

Hence, $\mathscr{A}$ cannot be a basic algebra.

As shown above, pseudo basic algebras are equivalent to bounded join-semilattices with sectional switching involutions. However, every section is a semilattice, i.e. it is again a bounded semilattice with sectional switching involutions. Hence, it can be converted into a pseudo basic algebra. How to organize its operations is shown in the following.

Theorem 7

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, let $\leq$ be its induced order and $p \in A$. The section $[p, 1]$ can be organized into a pseudo basic algebra $([p, 1]; \oplus_p, \neg_p, p)$ as follows:

$$\neg_p x = \neg x \oplus p \quad \text{and} \quad x \oplus_p y = \neg(\neg x \oplus p) \oplus y$$

for $x, y \in [p, 1]$.

Outline

Abstract – Skew basic algebras

Skew effect algebras were already introduced as a non-associative modification of the so-called effect algebras which serve as an algebraic axiomatization of the propositional logic of quantum mechanics. Since skew effect algebras have a partial binary operation, we search for an algebra with a total binary operation which extends a given skew effect algebra and such that the underlying posets coincide. It turns out that the suitable candidate is a skew basic algebra introduced in this section. Algebraic properties of skew basic algebras are described and they are compared with pseudo basic algebras.

Effect algebras were introduced by D.J. Foulis and M.K. Bennett as a tool for axiomatization of propositional logic of quantum mechanics. Their non-associative modification was defined by I. Chajda and H. Länger under the name skew effect algebras. It was shown that every skew effect algebra is in fact an ordered set with sectional switching involutions and an antitone global involution. This motivates us to describe this structure by a total algebra (with everywhere defined operations) similarly as it was done for effect algebras.

Recall that by a **skew effect algebra** is meant a partial algebra $\mathscr{S} = (S; +, ', 0, 1)$ of type $(2, 1, 0, 0)$ satisfying the following axioms:

(S1) if $x + y$ is defined then so is $y + x$ and $x + y = y + x$

(S2) $x + y = 1$ if and only if $y = x'$

(S3) if $x + 1$ is defined then $x = 0$

(S4) if $x + y = z$ then $x' = z' + y$

(S5) if $x' + (x + y)$ is defined then $y = 0$

(S6) if $(x + y) + z$ is defined then there exists an element $u \in S$ such that $(x + y) + z = x + u$.

The element x' is called a **supplement of** x.

It was shown that every effect algebra is a skew effect algebra and, moreover, a skew effect algebra is an effect algebra if and only if the partial operation $+$ is associative.

If $\mathscr{S} = (S; +, ', 0, 1)$ is a skew effect algebra and a binary relation $\leq$ is defined by

$$x \leq y \text{ if there exists } z \in S \text{ with } y = x + z$$

then $\leq$ is a partial order on S and 0 is the least and 1 the greatest element. It will be called an **induced order** of $\mathscr{S}$. A skew effect algebra $\mathscr{S}$ is called a **lattice skew effect algebra** if the induced ordered set $(S; \leq)$ is a lattice. Moreover, the mapping $x \mapsto x'$ is an antitone involution on S. Defining for $a \in S$ a mapping $x \mapsto x^a$ on the section $[a, 1]$ by $x^a = x' + a$, we have section switching involution. Since this section involution is defined for every element $a \in S$, we define the so-called **induced poset with section switching involutions** $(S; \leq, (^a)_{a \in S}, 0, 1)$, where $x^0 = x'$ is the so-called **global antitone involution**.

It was proved in that also conversely, if $(S; \leq, (^a)_{a \in S}, 0, 1)$ is such a poset and we define $x + y = (y^0)^x$ for $x \leq y^0$ then we get an induced skew effect algebra $\mathscr{A}(\mathscr{S})$. Moreover, these assignments of induced poset with section switching involutions and of a skew effect algebra $\mathscr{A}(\mathscr{S})$ are one-to-one correspondences.

Our aim is to define an algebra (with everywhere defined operations) whose induced ordered set would have the same properties as that of a skew effect algebra.

Definition 2

An algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is called a **skew basic algebra** if it satisfies the following axioms:

(A1) $x \oplus 0 = x$

(A2) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$

(A3) $x \oplus (\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus z) = 1$, where $1 = \neg 0$

(A4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(A5) $(\neg(x \oplus y) \oplus y) \oplus x = 1$

(A6) $x \oplus y = y \oplus \neg(\neg(x \oplus y) \oplus y)$.

Note that the axioms (A1)–(A6) are independent.

Theorem 8

The variety of skew basic algebras is congruence regular and arithmetical.

For the next, let us recall that an algebra $\mathscr{D} = (D; \sqcup)$ of type (2) is called a **commutative directoid** if it satisfies the axioms

(D1) $x \sqcup x = x$

(D2) $x \sqcup y = y \sqcup x$

(D3) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

If $\mathscr{D} = (D; \sqcup)$ is a commutative directoid and $\leq$ is a binary relation on D given as follows

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y$$

then $(D; \leq)$ is an ordered set which is **directed**, i.e. the set of upper bounds, so-called upper cone $U(a, b) = \{x \in D; a \leq x \text{ and } b \leq x\} \neq \emptyset$ for all $a, b \in D$. Also conversely, if $(D; \leq)$ is a directed ordered set and we define a binary operation $\sqcup$ as follows

- if $x \leq y$ then $x \sqcup y = y = y \sqcup x$
- if neither $x \leq y$ nor $y \leq x$ then $x \sqcup y = y \sqcup x \in U(x, y)$ is an arbitrary element from the upper cone $U(x, y)$

then $(D; \sqcup)$ is a commutative directoid. Let us note that every ordered set with a greatest element 1 is directed since $1 \in U(x, y)$ for all $x, y \in D$.

Theorem 9

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a skew basic algebra, $\leq$ its induced order. Define $1 = \neg 0$, $x \sqcup y = \neg(\neg x \oplus y) \oplus y$ and for $a \leq x$, $x^a = \neg x \oplus a$. Then $\mathscr{D}(A) = (A; \sqcup, (^a)_{a \in A}, 0, 1)$ is a bounded commutative directoid with sectional switching involutions where

(a) the global involution $x \mapsto x^0$ is antitone

(b) if $y \leq x$ then $x^y = (y^0)^{(x^0)}$.

Moreover, the order of the directoid $(A; \sqcup)$ coincides with that of $\mathscr{A}$.

In what follows, we can prove the converse.

Theorem 10

Let $\mathscr{D} = (D; \sqcup, (^a)_{a \in D}, 0, 1)$ be a bounded commutative directoid with sectional switching involutions such that the global involution $x \mapsto x^0$ is antitone and $y \leq x \Rightarrow x^y = (y^0)^{(x^0)}$. Define $x \oplus y = (x^0 \sqcup y)^y$ and $\neg x = x^0$. Then $\mathscr{A}(D) = (D; \oplus, \neg, 0)$ is a skew basic algebra.

Let us note that if $\mathscr{A} = (A; \oplus, \neg, 0)$ is a skew basic algebra and $\mathscr{D}(A)$ the assigned commutative directoid then $\mathscr{A}(\mathscr{D}(A)) = \mathscr{A}$.

Example 2 (1/2)

Let $A = \{0, a, b, c, d, 1\}$. Consider the skew basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	c	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
$\neg x$	1	d	c	b	a	0

.

Note that the operation $\oplus$ is commutative, i.e. $x \oplus y = y \oplus x$ for each $x, y \in A$.

Example 2 (2/2)

Its underlying poset is depicted in Figure 3. We have,
$a \sqcup b = b \sqcup a = \neg(\neg a \oplus b) \oplus b = d$ and $a \sqcup c = c$, $a \sqcup d = d$, $b \sqcup c = c$, $b \sqcup d = d$, $c \sqcup d = 1$ determining the commutative directoid $\mathscr{D}(A)$.

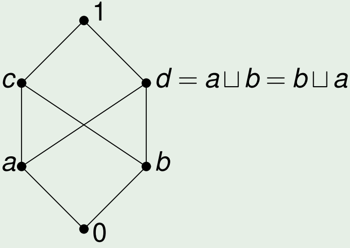


Figure 3

The sectional switching involutions are as follows:
In $[0,1]$, $x^0 = \neg x$.
In $[a,1]$ it is: $a^a = 1$, $c^a = \neg c \oplus a = d$, $d^a = \neg d \oplus a = c$, $1^a = a$.
In $[b,1]$ it is: $b^b = 1$, $c^b = \neg c \oplus b = c$, $d^b = \neg d \oplus b = d$, $1^b = b$.
In $[c,1], [d,1]$ and $[1,1]$ it is determined uniquely.
Note that here the involutions are antitone in each section.

Example 3 (1/2)

Consider the skew basic algebra $\mathscr{A} = (\{0, a, b, c, d, e, 1\}; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	e	c	d	d	1	1
b	b	c	d	e	1	e	1
c	c	d	e	1	1	1	1
d	d	a	1	1	1	1	1
e	e	1	b	1	1	1	1
1	1	1	1	1	1	1	1

x	0	a	b	c	d	e	1
$\neg x$	1	e	d	c	b	a	0

.

Note that the operation $\oplus$ is not commutative since e.g. $d \oplus a = a \neq d = a \oplus d$. The induced bounded commutative directoid $\mathscr{D}(A)$ is visualized in Figure 4.

Example 3 (2/2)

Let us note that this poset is a lattice with respect to suprema and infima but it is not a semilattice with respect to the directoid operation $\sqcup$ because $a \sqcup b = b \sqcup a = 1$ but $\sup(a, b) = c$.

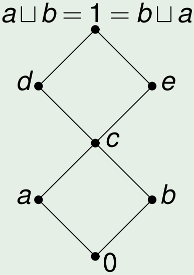


Figure 4

The sectional switching involutions are as follows:
In $[0, 1]$, $x^0 = \neg x$. In $[d, 1], [e, 1]$ and $[1, 1]$ it is determined uniquely.
In $[a, 1]$ it is: $a^a = 1$, $c^a = d$, $d^a = c$, $e^a = e$, $1^a = a$.
In $[b, 1]$ it is: $b^b = 1$, $c^b = e$, $d^b = d$, $e^b = c$, $1^b = b$.
In $[c, 1]$ it is: $c^c = 1$, $d^c = e$, $e^c = d$ and $1^c = c$.
Note that the involutions in sections $[a, 1]$ and $[b, 1]$ are not antitone.

A skew basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ will be called a **lattice skew basic algebra** if the induced poset $(A; \leq)$ is a lattice and $x \vee y = \neg(\neg x \oplus y) \oplus y$. Of course, since $x \mapsto x^0 = \neg x$ is an antitone involution, we get $x \wedge y = \neg(\neg x \vee \neg y)$ by DeMorgan laws.

Theorem 11

A skew basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is a lattice skew basic algebra if and only if it satisfies the axiom

(L) $\neg(\neg x \oplus (\neg(\neg y \oplus z) \oplus z)) \oplus (\neg(\neg y \oplus z) \oplus z) =$
$\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z.$

In the previous two examples we have seen directoids, which are not lattices. Obviously the identity (L) does not hold for them.

We finish this section with an interesting connection between skew basic algebras and MV-algebras.

Theorem 12

A skew basic algebra is an MV-algebra if and only if the operation $\oplus$ is associative.

Outline

Let $\mathscr{S} = (S; +, ', 0, 1)$ be a skew effect algebra and $\leq$ its induced order. The operation $x + y$ is defined if and only if the elements x, y are **orthogonal**, i.e. if $x \leq y'$ which is equivalent to $y \leq x'$. Then $x + y = (y')^x = (x')^y$ in the induced poset with sectional switching involutions.

Theorem 13

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a skew basic algebra and $\leq$ its induced order. Define $1 = \neg 0$, $x' = \neg x$ and

$$x + y = x \oplus y \quad \text{if and only if} \quad x \leq y'.$$

Then $\mathscr{S}(A) = (A; +, ', 0, 1)$ is a skew effect algebra whose induced order coincides with $\leq$.

Example 4 (1/2)

Let $A = \{0, a, b, c, d, 1\}$. Consider the skew basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	c	1	1
b	b	c	d	1	d	1
c	c	a	1	1	1	1
d	d	1	b	1	1	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
$\neg x$	1	d	c	b	a	0

.

Its underlying poset is depicted in Figure 5, $x \sqcup y = \neg(\neg x \oplus y) \oplus y$.

$a \sqcup b = 1 = b \sqcup a$

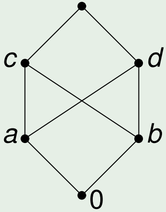


Figure 5

Example 4 (2/2)

Now, by the last theorem, we define $x' = \neg x$ and

$$x + y = x \oplus y \quad \text{if and only if} \quad x \leq y'.$$

Then $\mathscr{S}(A) = (A; +, ', 0, 1)$ is a skew effect algebra where the partial operation $+$ is given by the following table

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	–	1	–
b	b	c	d	1	–	–
c	c	–	1	–	–	–
d	d	1	–	–	–	–
1	1	–	–	–	–	–

Let us note that the partial operation $+$ in the skew effect algebra $\mathscr{S}(A)$ is commutative contrary to the fact that skew basic algebra $\mathscr{A}$ is not commutative because e.g. $a \oplus c \neq c \oplus a$.

Now, we are ready to describe a completion of a skew effect algebra into a total algebra which is a skew basic algebra.

Theorem 14

Let $\mathscr{S} = (S; +, ', 0, 1)$ be a skew effect algebra and $\leq$ its induced order. Then there exists a skew basic algebra $\mathscr{A}(S) = (S; \oplus, \neg, 0)$ such that $\leq$ coincides with the induced order of $\mathscr{S}$ and

$$x + y = x \oplus y \quad \text{if} \quad x \leq y'.$$

Moreover, if $\mathscr{S}$ is a lattice skew effect algebra then $\mathscr{A}(S)$ is a lattice skew basic algebra and the underlying lattices coincide.

As shown above, a skew basic algebra $\mathscr{A}(S)$ is assigned to a skew effect algebra in a unique way if and only if $\mathscr{S}$ is a lattice skew effect algebra. On the other hand, we have shown that the assignment $\mathscr{S} \mapsto \mathscr{A}(\mathscr{S})$ captures the whole information about $\mathscr{S}$ for any assigned $\mathscr{A}(\mathscr{S})$ because $x + y = x \oplus y$ whenever $x + y$ is defined.

Outline

Pseudo basic algebras cannot be used as extensions of skew effect algebras because orthogonal elements in pseudo basic algebras need not commute. Moreover, the underlying poset of a pseudo basic algebra is a semilattice which need not be the case of a skew effect algebra, see examples in the previous section. In what follows, we are going to show that every strict pseudo basic algebra can become a lattice skew basic algebra when the orthogonal elements commute.

In the example below, we get a strict pseudo basic algebra, which is not a skew basic algebra.

Example 5

Define $\oplus$ and $\neg$ on the set $\{0, a, b, c, 1\}$ by the following tables

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	c	c	1	1
b	b	b	1	1	1
c	c	1	1	1	1
1	1	1	1	1	1

x	0	a	b	c	1
$\neg x$	1	c	b	a	0

.

One can easily check that $\mathscr{A} = (\{0, a, b, c, 1\}; \oplus, \neg, 0)$ is a strict pseudo basic algebra.
On the other hand $\mathscr{A}$ does not satisfy the axiom (A6) because

$$b \oplus a = b \neq c = a \oplus b = a \oplus \neg(\neg(b \oplus a) \oplus a)$$

and hence it is not a skew basic algebra.

The axiom (A6) is just the missing condition characterizing lattice skew basic algebras among strict pseudo basic algebras.

Theorem 15

A strict pseudo basic algebra $\mathscr{A}$ is a lattice skew basic algebra if and only if it satisfies the identity (A6).

In what follows, we are checking if a section $[p,1]$ of a given skew basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ can be organized into a skew basic algebra again. Of course, it is not possible for every $p \in A$ because the sectional involution x^p need not be antitone and, in the section $[p,1]$, it should become a global involution. However, it is the only constrain as shown in the following.

Theorem 16

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a skew basic algebra and $a \in A$. On a section $[a,1]$ we define $x \oplus_a y = \neg(\neg x \oplus a) \oplus y$ and $\neg_a x = \neg x \oplus a$ for all $x, y \in [a,1]$. Then $([a,1]; \oplus_a, \neg_a, a)$ is a skew basic algebra if and only if the sectional involution x^a in $\mathscr{A}$ is antitone.

Example 6

Consider the skew basic algebra of Example 2. Then the sectional switching involutions in the nontrivial sections $[a, 1]$, $[b, 1]$ are antitone and hence these sections can be converted into skew basic algebras. On the contrary, in the skew basic algebra of Example 3, the sectional switching involutions in the nontrivial sections $[a, 1]$, $[b, 1]$ are not antitone and hence these sections cannot be organized in skew basic algebras.

Now, we show the number of non-isomorhic models of a given algebras of a given number of elements. Specifically, we focus on the MV-algebras, basic algebras (BA), pseudo basic algebras (PBA), commutative PBA, strict PBA, skew basic algebras (SBA) and commutative SBA. The numerical values in the following table were calculated using the program Prover9 and Mace4, see `http://www.cs.unm.edu/~mccune/mace4/`. The values in the fields marked with "–" values are not known, due to excessive computational complexity (time and/or memory).

	2	3	4	5	6	7	8	9	10
MV-algebras	1	1	2	1	2	1	3	2	2
BA	1	1	3	4	11	15	53	81	305
PBA	1	1	4	23	330	11516	–	–	–
com. PBA	1	1	2	1	2	2	5	3	5
strict PBA	1	1	3	5	25	164	4698	–	–
SBA	1	1	3	4	13	25	288	2116	–
com. SBA	1	1	2	1	3	2	17	6	69

The table columns correspond to sizes of a given algebras (number of their elements). The first row contains the numbers of non-isomorphic MV-algebras, . . .

We can see from the table, for instance, that:

- there exists a 4-element basic algebra which is not an MV-algebra and that there exists a 4-element pseudo basic algebra which is not a basic algebra. It is 4-element chain, where $0 < a < b < 1$, $\neg 0 = 1$, $\neg a = a$, $\neg b = b$ and $\neg 1 = 0$, whence the corresponding involution is switching, but not antitone.
- there are 11516 non-isomorphic 7-element pseudo basic algebras, but only 164 of them are strict, and only two of them are commutative. Moreover, one of this two commutative pseudo basic algebras is not an MV-algebra (see Figure 2).

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Thank you for your attention.