Semigroups of sets of n-ary functions

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Example

$n = 2$: binary Boolean operations

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
$(0,0)$	1	0	1	1	1	0	0	0
$(0,1)$	1	1	0	1	1	1	1	0
$(1,0)$	1	1	1	0	1	1	0	1
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	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
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- $T(A, B) := \{\alpha \in T(A) : im\alpha \subseteq B\}$

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- 2013: "Rank and idempotent rank of finite full transformation semigroups with restricted range", Worachead Sommanee,
- 2016: "On Semigroups of Orientation-preserving Transformations with Restricted Range", Vítor H. Fernandes, Preeyanuch Honyamb, Teresa M. Quinteiroc, and Boorapa Singhad

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Fact

$h : P_2^n \to T(\{0,1\}^n, \Delta_{\times n})$ with $h : f \mapsto \alpha_f$ is bijective.

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- This shows $h(f+g) = h(g) \cdot h(f)$.

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- $(SetP_2^{\times n}; \cup, *)$ forms a semiring
- $g : P_2^{\times n} \longrightarrow (SetP_2^{\times n}, *)$ with $a \longmapsto \{a\}$ is an embedding

Our Question

Problem

What about $(SetP_2^{\times n}, *)$?

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- although $\begin{pmatrix} \overline{13} & \overline{24} \\ 2 & 1 \end{pmatrix}$ is not idempotent

Idempotent subsemigroups

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Theorem

There are **exactly two** *maximal idempotent subsemigroups of* $(SetP_2^{\times n}, *)$.

Regular elements

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- A is regular since $A * A * A = A$ (i.e. $A = B$)

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Theorem

There are **exactly two** *maximal regular subsemigroups of $SetP_2^{\times n}$.*