Boolean Operations

- $n \in \mathbb{N}$
Boolean Operations

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- $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is called $n$-ary Boolean operation
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- \( P^n_2 \) set of all \( n \)-ary Boolean operations
- \( |P^n_2| = 2^{2^n} \)
Boolean Operations

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- $|P_2^n| = 2^{2^n}$

Example

$n = 2$: binary Boolean operations
<table>
<thead>
<tr>
<th></th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
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<tbody>
<tr>
<td>$(0,0)$</td>
<td>1</td>
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<td>$f_9$</td>
<td>$f_{10}$</td>
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<td>$f_{12}$</td>
<td>$f_{13}$</td>
<td>$f_{14}$</td>
<td>$f_{15}$</td>
<td>$f_{16}$</td>
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Definition of a binary operation $+$ on $P^2_2$ by
Binary operation

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<td>$f + g := S_n^n(f, g, \ldots, g)$, i.e. $(f + g)(\bar{a}) := f(g(\bar{a}), \ldots, g(\bar{a}))$ for all $\bar{a} \in {0, 1}^n$</td>
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**Example**

- $f_3 + f_4 = f_1$ since
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- $f_3(f_4(0, 0), f_4(0, 0)) = f_3(1, 1) = 1$
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Semigroup

$(P^n_2, +)$ is a semigroup
Semigroup

• $\left( P_n^2, + \right)$ is a semigroup

Fact

The semigroup $\left( P_n^2, + \right)$ is well studied in Semigroup Theory.
Semigroup

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Fact

The semigroup \((P^n_2, +)\) is well studied in Semigroup Theory.

- Why?
Let \( A \) be any set
Transformations

- Let $A$ be any set
- $\alpha : A \rightarrow A$ is called \textbf{transformation on} $A$
Let $A$ be any set

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- $\text{im}\alpha := \{a\alpha : a \in A\}$ is called **image of** $\alpha$
- $B \subseteq A$
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$\text{im}\alpha := \{a\alpha : a \in A\}$ is called \textbf{image of} $\alpha$

$B \subseteq A$

$T(A, B) := \{\alpha \in T(A) : \text{im}\alpha \subseteq B\}$
Transformations with restricted range

- $T(A, B) \leq T(A)$
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- 2016: "On Semigroups of Orientation-preserving Transformations with Restricted Range", Vítor H. Fernandes, Preeyanuch Honyamb, Teresa M. Quinteiroc, and Boorapa Singhad
A = \{0, 1\}^n
A bijection

- $A = \{0, 1\}^n$
- $B = \Delta \times n := \{(0, \ldots, 0), (1, \ldots, 1)\} \subseteq \{0, 1\}^n$
\begin{itemize}
\item $A = \{0, 1\}^n$
\item $B = \Delta \times n := \{(0, \ldots, 0), (1, \ldots, 1)\} \subseteq \{0, 1\}^n$
\end{itemize}

**Definition**

Let $f \in P_2^n$, $\alpha_f : \{0, 1\}^n \rightarrow \Delta \times n$ be defined by $\overline{a}\alpha_f := (f(\overline{a}), \ldots, f(\overline{a}))$. 

\[ f \in P_2^n, \alpha_f : \{0, 1\}^n \rightarrow \Delta \times n \text{ by } \overline{a}\alpha_f := (f(\overline{a}), \ldots, f(\overline{a})). \]
Bijection

- \( A = \{0, 1\}^n \)
- \( B = \Delta \times n := \{(0, \ldots, 0), (1, \ldots, 1)\} \subseteq \{0, 1\}^n \)

**Definition**

\( f \in P_2^n, \alpha_f : \{0, 1\}^n \rightarrow \Delta \times n \) by \( \overline{a}\alpha_f := (f(\overline{a}), \ldots, f(\overline{a})) \).

\( \{\alpha_f : f \in P_2^n\} \subseteq T(\{0, 1\}^n, \Delta \times n) \)
Bijection

- $A = \{0, 1\}^n$
- $B = \Delta \times n := \{(0, \ldots, 0), (1, \ldots, 1)\} \subseteq \{0, 1\}^n$

**Definition**

$f \in P^n_2$, $\alpha_f : \{0, 1\}^n \to \Delta \times n$ by $\bar{a}\alpha_f := (f(\bar{a}), \ldots, f(\bar{a}))$.  

- $\{\alpha_f : f \in P^n_2\} \subseteq T(\{0, 1\}^n, \Delta \times n)$

**Fact**

$h : P^n_2 \to T(\{0, 1\}^n, \Delta \times n)$ with $h : f \mapsto \alpha_f$ is bijective.
We can show that \( h \) is isomorphism.
Isomorphism

- We can show that $h$ is isomorphism

**Theorem**

$$\left( P_n^2; + \right) \cong T(\{0,1\}^n, \Delta \times n).$$
We can show that $h$ is isomorphism.

**Theorem**

$$(P^n_2; +) \cong T(\{0, 1\}^n, \Delta \times n).$$

**Proof.**
Isomorphism

We can show that $h$ is isomorphism

**Theorem**

$\left( P_2^n ; + \right) \cong T(\{0, 1\}^n, \Delta \times_n)$.

**Proof.**

- Let $h : P_2^n \to T(\{0, 1\}^n, \Delta \times_n)$ with $h : f \mapsto \alpha_f$
Isomorphism

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**Theorem**

$(P_2^n; +) \cong T([0, 1]^n, \Delta \times n)$.

**Proof.**

- Let $h : P_2^n \rightarrow T([0, 1]^n, \Delta \times n)$ with $h : f \mapsto \alpha_f$
- $h$ is bijective
Isomorphism

- We can show that $h$ is isomorphism

**Theorem**

$$(P_2^n; +) \cong T(\{0, 1\}^n, \Delta_{\times n}).$$

**Proof.**

- Let $h : P_2^n \rightarrow T(\{0, 1\}^n, \Delta_{\times n})$ with $h : f \mapsto \alpha_f$
- $h$ is bijective
- $\bar{a}(h(f + g)) = \bar{a}\alpha(f + g)$
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- $\bar{a}(h(f + g)) = \bar{a}\alpha(f + g)$
- $= (f + g(\bar{a}), \ldots, f + g(\bar{a}))$
Isomorphism

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**Theorem**

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- Let $h : P_2^n \rightarrow T(\{0, 1\}^n, \Delta_{\times n})$ with $h : f \mapsto \alpha_f$
- $h$ is bijective
- $\overline{a}(h(f + g)) = \overline{a}\alpha_{f+g}$
  
  $= (f + g(\overline{a}), \ldots, f + g(\overline{a}))$
  
  $= (f(g(\overline{a}), \ldots, g(\overline{a})), \ldots, f(g(\overline{a}), \ldots, g(\overline{a})))$
  
  $= (g(\overline{a}), \ldots, g(\overline{a}))\alpha_f = (\overline{a}\alpha_g)\alpha_f = \overline{a}(\alpha_g\alpha_f)$
Isomorphism

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- $= (g(\bar{a}), \ldots, g(\bar{a}))\alpha_f = (\bar{a}\alpha_g)\alpha_f = \bar{a}(\alpha_g\alpha_f)$
- $= \bar{a}(h(g)h(f)).$
We can show that $h$ is isomorphism.

**Theorem**

$$(P^n_2; +) \cong T(\{0, 1\}^n, \Delta \times n).$$

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- Let $h : P^n_2 \rightarrow T(\{0, 1\}^n, \Delta \times n)$ with $h : f \mapsto \alpha_f$
- $h$ is bijective
- $\bar{a}(h(f + g)) = \bar{a}(f + g)$
- $= (f + g(\bar{a}), \ldots, f + g(\bar{a}))$
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- $= (g(\bar{a}), \ldots, g(\bar{a}))\alpha_f = (\bar{a}\alpha g)\alpha_f = \bar{a}(\alpha g \alpha f)$
- $= \bar{a}(h(g)h(f))$.
- This shows $h(f + g) = h(g) \cdot h(f)$. 
New approach

- $|\{0,1\}^n| = 2^n$, $X_{2^n} := \{1, 2, \ldots, 2^n\}$
New approach

- $|\{0, 1\}^n| = 2^n$, $X_{2^n} := \{1, 2, \ldots, 2^n\}$
- $T(\{0, 1\}^n, \Delta \times n) \cong T(X_{2^n}, \{1, 2\})$
New approach

- $|\{0,1\}^n| = 2^n$, $X_{2^n} := \{1, 2, \ldots, 2^n\}$
- $T(\{0,1\}^n, \Delta_{\times n}) \cong T(X_{2^n}, \{1, 2\})$
- $P_{2^{\times n}} := T(X_{2^n}, \{1, 2\})$
New approach

- $|\{0, 1\}^n| = 2^n$, $X_2^n := \{1, 2, \ldots, 2^n\}$
- $T(\{0, 1\}^n, \Delta \times n) \cong T(X_2^n, \{1, 2\})$
- $P_2^\times n := T(X_2^n, \{1, 2\})$
- $SetP_2^\times n$ set of all non-empty subsets of $P_2^\times n$
New approach

- $|\{0,1\}^n| = 2^n$, $X_{2^n} := \{1, 2, \ldots, 2^n\}$
- $T(\{0,1\}^n, \Delta \times_n) \cong T(X_{2^n}, \{1, 2\})$
- $P_{2^n} := T(X_{2^n}, \{1, 2\})$
- $SetP_{2^n}$ set of all non-empty subsets of $P_{2^n}$
- $|SetP_{2^n}| = 2^{\left|P_{2^n}\right|} - 1$
New approach

- $|\{0, 1\}^n| = 2^n$, $X_{2^n} := \{1, 2, \ldots, 2^n\}$
- $T(\{0, 1\}^n, \Delta \times n) \cong T(X_{2^n}, \{1, 2\})$
- $P_{2^n}^\times := T(X_{2^n}, \{1, 2\})$
- $SetP_{2^n}^\times$ set of all non-empty subsets of $P_{2^n}^\times$
- $|SetP_{2^n}^\times| = 2^{|P_{2^n}^\times|} - 1$

Example

$(n = 2) |SetP_2^\times| = 2^{|P_2^\times|} - 1 = 2^{16} - 1$
New approach

- $|\{0, 1\}^n| = 2^n$, $X_{2n} := \{1, 2, \ldots, 2^n\}$
- $T(\{0, 1\}^n, \Delta \times n) \cong T(X_{2n}, \{1, 2\})$
- $P_{2 \times n} := T(X_{2n}, \{1, 2\})$
- $\text{Set}P_{2 \times n}$ set of all non-empty subsets of $P_{2 \times n}$
- $|\text{Set}P_{2 \times n}| = 2|P_{2 \times n}| - 1$

Example

$(n = 2)$ $|\text{Set}P_{2 \times 2}| = 2|P_{2 \times 2}| - 1 = 2^{16} - 1$

- We introduce a **binary operation** $\ast$ on $\text{Set}P_{2 \times 2}$ by $A \ast B := \{ab : a \in A, b \in B\}$
New approach

- \(|\{0,1\}^n| = 2^n, X_2^n := \{1, 2, \ldots, 2^n\}\)
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- \(P_2^{\times n} := T(X_2^n, \{1, 2\})\)
- \(\text{Set}P_2^{\times n}\) set of all non-empty subsets of \(P_2^{\times n}\)
- \(|\text{Set}P_2^{\times n}| = 2^{|P_2^{\times n}|} - 1\)

Example

\((n = 2)\) \(|\text{Set}P_2^{\times 2}| = 2^{|P_2^{\times 2}|} - 1 = 2^{16} - 1\)

- We introduce a **binary operation** \(\ast\) on \(\text{Set}P_2^{\times 2}\) by \(A \ast B := \{ab : a \in A, b \in B\}\)
- \((\text{Set}P_2^{\times n}, \ast)\) forms a semigroup
New approach

- $|\{0, 1\}^n| = 2^n$, $X_2^n := \{1, 2, \ldots, 2^n\}$
- $T(\{0, 1\}^n, \Delta \times n) \cong T(X_2^n, \{1, 2\})$
- $P_2^\times n := T(X_2^n, \{1, 2\})$
- $SetP_2^\times n$ set of all non-empty subsets of $P_2^\times n$
- $|SetP_2^\times n| = 2|P_2^\times n| - 1$

Example

$(n = 2)\; |SetP_2^\times 2| = 2|P_2^\times 2| - 1 = 2^{16} - 1$

- We introduce a binary operation $\ast$ on $SetP_2^\times 2$ by $A \ast B := \{ab : a \in A, b \in B\}$
- $(SetP_2^\times n, \ast)$ forms a semigroup
- $(SetP_2^\times n; \cup, \ast)$ forms a semiring
New approach

- \(|\{0,1\}^n| = 2^n, X_2^n := \{1, 2, \ldots, 2^n\}\)
- \(T(\{0,1\}^n, \Delta \times_n) \cong T(X_2^n, \{1, 2\})\)
- \(P_2^\times_n := T(X_2^n, \{1, 2\})\)
- \(SetP_2^\times_n\) set of all non-empty subsets of \(P_2^\times_n\)
- \(|SetP_2^\times_n| = 2|P_2^\times_n| - 1\)

Example

\((n = 2)\) \( |SetP_2^\times_2| = 2|P_2^\times_2| - 1 = 2^{16} - 1\)

- We introduce a **binary operation** \(*\) on \(SetP_2^\times_2\) by
  \(A * B := \{ab : a \in A, b \in B\}\)
- \((SetP_2^\times_n, *)\) forms a semigroup
- \((SetP_2^\times_n; \cup, *)\) forms a semiring
- \(g : P_2^\times_n \rightarrow (SetP_2^\times_n, *)\) with \(a \mapsto \{a\}\) is an embedding
Our Question

Problem

What about \((\text{Set}P_2^n, \ast)\)?
A ∈ SetP_{2}^{\times n} is called idempotent if A ∗ A = A
Idempotent elements

- $A \in \text{SetP}_2^\times n$ is called **idempotent** if $A \ast A = A$
- $E(\text{SetP}_2^\times n)$ set of all idempotents in $\text{SetP}_2^\times n$
Idempotent elements

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**Example**

- $n = 2$
Idempotent elements

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- $E(\text{SetP}_2^\times n)$ set of all idempotents in $\text{SetP}_2^\times n$

**Example**

- $n = 2$
- \[ \left\{ \begin{pmatrix} 13 & 24 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 13 & 24 \\ 1 & 2 \end{pmatrix} \right\} \in E(\text{SetP}_2^\times 2) \]
Idempotent elements

- $A \in SetP_{2}^{\times n}$ is called **idempotent** if $A \ast A = A$
- $E(SetP_{2}^{\times n})$ set of all idempotents in $SetP_{2}^{\times n}$

**Example**

- $n = 2$
- \[
\left\{ \left( \begin{array}{cc} 13 & 24 \\ 2 & 1 \end{array} \right), \left( \begin{array}{cc} 13 & 24 \\ 1 & 2 \end{array} \right) \right\} \in E(SetP_{2}^{\times 2})
\]
- although $\left( \begin{array}{cc} 13 & 24 \\ 2 & 1 \end{array} \right)$ is not idempotent
Idempotent subsemigroups

- $S \leq (\text{Set}P_2^\times n, \ast)$ is called **idempotent subsemigroup** (or **band**) if $S \subseteq E(\text{Set}P_2^\times n)$.
Idempotent subsemigroups

- $S \subseteq (\text{Set}P_2^\times n, \ast)$ is called idempotent subsemigroup (or band) if $S \subseteq E(\text{Set}P_2^\times n)$.
- $(\text{Set}P_2^\times n, \ast)$ is not a band.
Idempotent subsemigroups

- \( S \leq (\text{SetP}_2 \times^n, \ast) \) is called **idempotent subsemigroup** (or **band**) if \( S \subseteq E(\text{SetP}_2 \times^n) \).
- \( (\text{SetP}_2 \times^n, \ast) \) is not a band.
- An idempotent subsemigroup \( S \) is called **maximal idempotent subsemigroup** if

\[
S \leq T \iff S = T \text{ or } T \not\subseteq E(\text{SetP}_2 \times^n).
\]
Idempotent subsemigroups

- $S \leq (\text{SetP}_2^\times n, \ast)$ is called idempotent subsemigroup (or band) if $S \subseteq E(\text{SetP}_2^\times n)$.
- $(\text{SetP}_2^\times n, \ast)$ is not a band.
- An idempotent subsemigroup $S$ is called maximal idempotent subsemigroup if

\[ S \leq T \implies S = T \text{ or } T \not\subseteq E(\text{SetP}_2^\times n). \]

Theorem

There are exactly two maximal idempotent subsemigroups of $(\text{SetP}_2^\times n, \ast)$. 
A ∈ $SetP_2^\times n$ is called **regular** if there is $B ∈ SetP_2^\times n$ with $A * B * A = A$.
Regular elements

- $A \in \text{Set}P_2^{\times n}$ is called regular if there is $B \in \text{Set}P_2^{\times n}$ with $A \ast B \ast A = A$.

**Fact**

*The idempotent elements are regular.*
A ∈ SetP_2^{×n} is called regular if there is B ∈ SetP_2^{×n} with A * B * A = A.

Fact

The idempotent elements are regular.

Example
Regular elements

- \( A \in Set_{P_2}^{\times n} \) is called **regular** if there is \( B \in Set_{P_2}^{\times n} \) with \( A * B * A = A \).

**Fact**

*The idempotent elements are regular.*

**Example**

- \( n = 2 \)
Regular elements

- $A \in \text{SetP}_2^{\times n}$ is called \textbf{regular} if there is $B \in \text{SetP}_2^{\times n}$ with $A \ast B \ast A = A$.

**Fact**

\textit{The idempotent elements are regular.}

**Example**

- $n = 2$
- $A = \left\{ \begin{pmatrix} 13 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2 \end{pmatrix} \right\}$
Regular elements

- $A \in \text{SetP}_2^{\times n}$ is called **regular** if there is $B \in \text{SetP}_2^{\times n}$ with $A \ast B \ast A = A$.

**Fact**

*The idempotent elements are regular.*

**Example**

- $n = 2$
- $A = \left\{ \begin{pmatrix} 13 & 24 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2 \end{pmatrix} \right\}$
- $A$ is not idempotent since $A \ast A = \left\{ \begin{pmatrix} 13 & 24 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1234 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2 \end{pmatrix} \right\}$
Regular elements

- \( A \in \text{Set}P_2^{\times n} \) is called **regular** if there is \( B \in \text{Set}P_2^{\times n} \) with \( A \ast B \ast A = A \).

**Fact**

*The idempotent elements are regular.*

**Example**

- \( n = 2 \)
- \( A = \left\{ \begin{pmatrix} 13 & 24 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2 \end{pmatrix} \right\} \)
- \( A \) is not idempotent since \( A \ast A = \left\{ \begin{pmatrix} 13 & 24 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1234 \\ 1 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2 \end{pmatrix} \right\} \)
- \( A \) is regular since \( A \ast A \ast A = A \) (i.e. \( A = B \))
A semigroup is called **regular** if all its elements are regular.
A semigroup is called **regular** if all its elements are regular.

**Fact**

$\text{SetP}_2^\times n$ is not regular
A semigroup is called **regular** if all its elements are regular.

**Fact**

\( \text{SetP}_2^\times n \text{ is not regular} \)

A regular subsemigroup \( S \) is called **maximal regular subsemigroup** if

\[ S \leq T \iff S = T \text{ or } T \text{ is not regular.} \]
A semigroup is called **regular** if all its elements are regular.

**Fact**

Set$P_{2}^{\times n}$ is not regular

A regular subsemigroup $S$ is called **maximal regular subsemigroup** if

$$S \leq T \iff S = T \text{ or } T \text{ is not regular.}$$

**Theorem**

*There are exactly two* maximal regular subsemigroups of $SetP_{2}^{\times n}$. 