Convex congruences

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1. Motivation

Motivation

Definition 1

- BCK-*algebra* = *algebra* $(A, \to, 1)$ *of type* $(2, 0)$ *satisfying*
  - $(A, \leq)$ *is a poset* $(x \leq y :\Leftrightarrow x \to y = 1)$
  - $x \to 1 \approx 1$
  - $1 \to x \approx x$
  - $x \to ((x \to y) \to y) \approx 1$
  - $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$
- BCI-*algebra* = *algebra* $(A, \to, 1)$ *of type* $(2, 0)$ *satisfying*
  - $(A, \leq)$ *is a poset* $(x \leq y :\Leftrightarrow x \to y = 1)$
  - $1 \to x \approx x$
  - $x \to ((x \to y) \to y) \approx 1$
  - $(y \to z) \to ((x \to y) \to (x \to z)) \approx 1$

Definition 2

$(A, \leq)$: *poset* $\wedge\ \Theta \in \operatorname{Equ} A \Rightarrow$

$$\Rightarrow (\Theta \textit{ convex } :\Leftrightarrow (a, b, c \in A \wedge a \leq b \leq c \wedge a \,\Theta\, c \Rightarrow a \,\Theta\, b))$$

Motivation, continued

Remark 3

BCK- *resp.* BCI-*algebras form a proper quasivariety and are therefore not closed under the formation of quotients.*

Theorem 4

- (*cf.* [2]) $\mathbf{A}$ BCK-*algebra* $\wedge\ \Theta \in \mathrm{Con}\,\mathbf{A} \Rightarrow$
  $\Rightarrow (\mathbf{A}/\Theta$ BCK-*algebra* $\Leftrightarrow \Theta$ *convex*$)$
- $\mathbf{A}$ BCI-*algebra* $\wedge\ \Theta \in \mathrm{Con}\,\mathbf{A} \Rightarrow$
  $\Rightarrow (\mathbf{A}/\Theta$ BCI-*algebra* $\Leftrightarrow \Theta$ *convex*$)$

2. Preliminaries

Preliminaries

- $\mathbf{A} = (A, \to, 1)$: algebra of type $(2, 0)$
- $a, b, c \in A$
- $\Theta \in \operatorname{Con} \mathbf{A}$

Definition 5

- $a \leq b :\Leftrightarrow a \to b = 1$
- $[a]\Theta \leq' [b]\Theta :\Leftrightarrow [a]\Theta \to [b]\Theta = [1]\Theta$

Remark 6

- $a \leq b \Rightarrow [a]\Theta \leq' [b]\Theta$
- $\leq$ *reflexive* $\Rightarrow \leq'$ *reflexive*

Properties and identities

Definition 7

$$(x, y \in A \textit{ and } [x]\Theta \leq' [y]\Theta) \;\Rightarrow\; \exists z \in [x]\Theta : z \leq y \quad (1)$$

$$(x, y \in A \textit{ and } [x]\Theta \leq' [y]\Theta) \;\Rightarrow\; \exists z \in [y]\Theta : x \leq z \quad (2)$$

Definition 8

$$x \to 1 \approx 1 \quad (3)$$

$$1 \to x \approx x \quad (4)$$

$$x \to ((x \to y) \to y) \approx 1 \quad (5)$$

$$(x \to y) \to ((y \to z) \to (x \to z)) \approx 1 \quad (6)$$

$$(y \to z) \to ((x \to y) \to (x \to z)) \approx 1 \quad (7)$$

Examples and relations between properties

Example 9

- **A** BCK-*algebra* $\Leftrightarrow (A, \leq)$ *poset* $\wedge$ (3) $\wedge$ (4) $\wedge$ (5) $\wedge$ (6)
- **A** BCI-*algebra* $\Leftrightarrow (A, \leq)$ *poset* $\wedge$ (4) $\wedge$ (5) $\wedge$ (7)

Lemma 10

- (2) $\not\Rightarrow$ (5)
- (4) $\wedge$ (5) $\Rightarrow$ (2)
- (4) $\wedge$ (5) $\not\Rightarrow$ (6)
- (4) $\wedge$ (5) $\not\Rightarrow$ (7)
- (4) $\wedge$ (6) $\Rightarrow (A, \leq)$ *quoset*
- (4) $\wedge$ (7) $\Rightarrow (A, \leq)$ *quoset*

3. Integral residuated posets

Integral residuated posets

Definition 11

$(A, \leq, \cdot, \rightarrow, 1)$ *integral residuated poset* $:\Leftrightarrow$

- $(A, \leq)$: *poset*
- $(A, \cdot, \rightarrow, 1)$: *algebra of type* $(2, 2, 0)$
- $(A, \cdot, 1)$: *commutative groupoid with neutral element* 1
- $x \leq 1$
- $xy \leq z \Leftrightarrow x \leq y \rightarrow z$.

Lemma 12

$(A, \leq, \cdot, \rightarrow, 1)$ *integral residuated poset* $\wedge\ a, b \in A \Rightarrow$

- $a \leq b \Leftrightarrow a \rightarrow b = 1$
- $a \leq b \Leftrightarrow a \leq 1 \rightarrow b$
- $(4) \wedge (5)$

Example of an integral residuated posets neither being a BCK- nor a BCI-algebra

Example 13

$A = \{a, b, c, 1\}$, $a < b < c < 1$

$\cdot$	a	b	c	1
a	a	a	a	a
b	a	a	b	b
c	a	b	b	c
1	a	b	c	1

$\to$	a	b	c	1
a	1	1	1	1
b	b	1	1	1
c	a	c	1	1
1	a	b	c	1

$\Rightarrow \mathbf{A} = (A, \leq, \cdot, \to, 1)$ *integral residuated poset*

Example 13, continued

Example 14

- $(c \to b) \to ((b \to a) \to (c \to a)) = c \to (b \to a) = c \to b = c \neq 1 \Rightarrow \neg(6) \Rightarrow$ **A** *not a* BCK-*algebra*
- $(b \to a) \to ((c \to b) \to (c \to a)) = b \to (c \to a) = b \to a = b \neq 1 \Rightarrow \neg(7) \Rightarrow$ **A** *not a* BCI-*algebra*

Lemma 15

$(A, \leq, \cdot, \to, 1)$ *integral residuated poset* $\wedge\ \Theta \in \mathrm{Con}(A, \cdot, \to, 1) \Rightarrow (1)$

4. Convex congruences

Convex congruences

Lemma 16

$\leq'$ *antisymmetric* $\Rightarrow \Theta$ *convex*

Theorem 17

- $(A/\Theta, \leq')$ *poset* $\Rightarrow \Theta$ *convex*
- $(A, \leq)$ *poset* $\not\Rightarrow \Theta$ *convex*

$(A, \leq)$ poset $\not\Rightarrow \Theta$ convex

Example 18

- $A = \{a, b, 1\}$
-

$\rightarrow$	a	b	1
a	1	1	1
b	a	1	1
1	a	a	1

- $\mathbf{A} = (A, \rightarrow, 1)$
- $\Theta = \{a, 1\}^2 \cup \{b\}^2$
- $a < b < 1$
- Θ *not convex*
- $\{b\} \leq' \{a, 1\} \leq' \{b\} \Rightarrow \leq'$ *not antisymmetric*

Characterization of convexity

Theorem 19

- $(A, \leq)$ *poset* $\wedge$ (2) $\Rightarrow ((A/\Theta, \leq')$ *poset* $\Leftrightarrow \Theta$ *convex*)
- $(A, \leq)$ *poset* $\wedge$ Θ *convex* $\not\Rightarrow (A/\Theta, \leq')$ *poset*

Example 20

- $A = (\{a, b, 1\}$
$\to$	a	b	1
a	1	1	1
b	b	1	1
1	b	b	1
- $\Theta = \{a\}^2 \cup \{b, 1\}^2$, $a < b < 1$, Θ *convex*
- $\{a\} \leq' \{b, 1\} \leq' \{a\} \Rightarrow (A/\Theta, \leq')$ *not a poset*
- $[1]\Theta \leq' [a]\Theta \wedge 1 \not\leq a \Rightarrow \neg(2)$

Characterizations of convexity, continued

Theorem 21

$(A, \leq)$ *poset* $\wedge$ $(1) \Rightarrow ((A/\Theta, \leq')$ *poset* $\Leftrightarrow \Theta$ *convex*$)$

Theorem 22

$(A, \leq, \cdot, \rightarrow, 1)$ *integral residuated poset* $\wedge\ \Theta \in \mathrm{Con}(A, \cdot, \rightarrow, 1) \Rightarrow$
$\Rightarrow ((A/\Theta, \leq', \cdot, \rightarrow, [1]\Theta)$ *integral residuated poset* $\Leftrightarrow \Theta$ *convex*$)$

Theorem 23

- $\leq$ *antisymmetric* $\wedge\ (4) \wedge ((6) \vee (7)) \Rightarrow (A, \leq)$ *poset*
- $(4) \wedge (5) \wedge ((6) \vee (7)) \Rightarrow ((A/\Theta, \leq')$ *poset* $\Leftrightarrow \Theta$ *convex*$)$

Corollary 24

- (*cf.* [2]) $\mathbf{A}$ BCK-*algebra* $\Rightarrow (\mathbf{A}/\Theta$ BCK-*algebra* $\Leftrightarrow \Theta$ *convex*$)$
- $\mathbf{A}$ BCI-*algebra* $\Rightarrow (\mathbf{A}/\Theta$ BCI-*algebra* $\Leftrightarrow \Theta$ *convex*$)$

5. References

References

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Thank you for your attention!