Permutation classes closed under pattern involvement and composition

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joint work with
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54th Summer School on General Algebra and Ordered Sets
(SSAOS 2016)
Trojanovice, 3–9 September 2016
We consider permutations of \( \{1, \ldots, n\} \), for some \( n \in \mathbb{N}_+ \).

On the one hand, a permutation is a bijective map on \( \{1, \ldots, n\} \).

On the other hand, a permutation \( \pi \in S_n \) will be considered as a string of length \( n \):

\[
\pi = \pi_1 \pi_2 \ldots \pi_n,
\]

where \( \pi_i = \pi(i) \).
\begin{align*}
\sigma = \sigma_1 \ldots \sigma_\ell \in S_\ell & \quad \tau = \tau_1 \ldots \tau_\ell \in S_n \quad (\ell \leq n)
\end{align*}

\(\sigma\) is a \textbf{pattern} of \(\tau\) (or \(\tau\) \textbf{involves} \(\sigma\)), in symbols, \(\sigma \leq \tau\), if there exists a substring \(\tau_{i_1} \ldots \tau_{i_\ell}\) \((i_1 < i_2 < \cdots < i_\ell)\) that is order-isomorphic to \(\sigma_1 \ldots \sigma_\ell\).

\textbf{Example}

\[
\begin{array}{cccc}
5 & 13 & 26 & 74 \\
3 & 14 & 2 & 67 & 4
\end{array}
\]

\[
3142 \leq 5132674
\]
Permutation patterns

\[ \sigma = \sigma_1 \ldots \sigma_\ell \in S_\ell \quad \tau = \tau_1 \ldots \tau_\ell \in S_n \quad (\ell \leq n) \]

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**Example**

\[
\begin{array}{cccc}
5 & 1 & 3 & 2 \\
1 & 3 & 4 & 2 \\
3 & 1 & 4 & 2 \\
\end{array}
\]

\[3142 \leq 5132674\]
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**Example**

\[
\begin{array}{cccc}
5 & 1 & 3 & 2 \\
5 & 2 & 6 & 4 \\
3 & 1 & 4 & 2 \\
\end{array}
\]

\( 3142 \leq 5132674 \)
\[ \sigma = \sigma_1 \ldots \sigma_\ell \in S_\ell \quad \tau = \tau_1 \ldots \tau_\ell \in S_n \quad (\ell \leq n) \]

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\]
σ = σ₁ \ldots σ_ℓ ∈ S_ℓ \quad \tau = \tau₁ \ldots \tau_ℓ ∈ S_n \quad (ℓ ≤ n)

σ is a **pattern** of \( \tau \) (or \( \tau \) **involves** \( σ \)), in symbols, \( σ ≤ τ \),
if there exists a substring \( \tau_{i₁} \ldots \tau_{i_ℓ} \) \((i₁ < i₂ < \cdots < i_ℓ)\) that is
order-isomorphic to \( σ₁ \ldots σ_ℓ \).

\( \tau \) **avoids** \( σ \) if \( σ \not\leq \ τ \).

The pattern involvement relation \( ≤ \) is a partial order on the set
\( \mathbb{P} := \bigcup_{n≥1} S_n \) of all finite permutations. Downward closed subsets of \( \mathbb{P} \)
under \( ≤ \) are called **permutation classes**.
Permutation patterns

\[ S_n \]

\[ S_\ell \]

\[ S_4 \]

\[ S_3 \]

\[ S_2 \]

\[ S_1 \]

\[ \sigma \leq \tau \]
Permutation patterns

\[ \text{Pat}^{(\ell)}_\tau = \{ \sigma \in S_\ell \mid \sigma \preceq \tau \} \]
Permutation patterns

\[ \text{Pat}^{(\ell)} T = \bigcup_{\tau \in T} \text{Pat}^{(\ell)} \tau = \{ \sigma \in S_\ell \mid \exists \tau \in T: \sigma \leq \tau \} \]
Permutation patterns

\[ \text{Comp}^{(n)} S = \{ \tau \in S_n \mid \text{Pat}^{(\ell)} \tau \subseteq S \} \]

\[ \text{Pat}^{(\ell)} T = \bigcup_{\tau \in T} \text{Pat}^{(\ell)} \tau = \{ \sigma \in S_\ell \mid \exists \tau \in T : \sigma \leq \tau \} \]
The operators $\text{Pat}^{(\ell)}$ and $\text{Comp}^{(n)}$ constitute a monotone Galois connection between $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$. This is in fact the monotone Galois connection induced by the pattern avoidance relation $\not\preceq$ between $S_\ell$ and $S_n$.

$S \subseteq S_\ell, \ T \subseteq S_n \ (\ell \leq n)$

$\text{Comp}^{(n)} S := \left\{ \tau \in S_n \mid \text{Pat}^{(\ell)} \tau \subseteq S \right\} = \left\{ \tau \in S_n \mid \forall \sigma \in S_\ell \setminus S : \sigma \not\preceq \tau \right\}$,

$\text{Pat}^{(\ell)} T := \bigcup_{\tau \in T} \text{Pat}^{(\ell)} \tau = S_\ell \setminus \left\{ \sigma \in S_\ell \mid \forall \tau \in T : \sigma \not\preceq \tau \right\}$.

Closures and kernels:

$\text{Pat}^{(\ell)} \text{Comp}^{(n)} S \subseteq S$,

$T \subseteq \text{Comp}^{(n)} \text{Pat}^{(\ell)} T$,

$\text{Comp}^{(n)} S = \text{Comp}^{(n)} \text{Pat}^{(\ell)} \text{Comp}^{(n)} S$,

$\text{Pat}^{(\ell)} T = \text{Pat}^{(\ell)} \text{Comp}^{(n)} \text{Pat}^{(\ell)} T$. 
Further details on this Galois connection and the related Galois connection between the subgroup lattices of $S_\ell$ and $S_n$ in

E. Lehtonen, R. Pöschel,
The group case for $\text{Comp}^{(n)} S$

**Proposition**

If $S$ is a subgroup of $S_\ell$, then $\text{Comp}^{(n)} S$ is a subgroup of $S_n$.

**Sketch of a proof.**

Assume that $S \leq S_\ell$. Let $\pi, \tau \in \text{Comp}^{(n)} S$. Thus $\text{Pat}^{(\ell)} \pi, \text{Pat}^{(\ell)} \tau \subseteq S$.

It holds that

$$\text{Pat}^{(\ell)} \pi^{-1} = (\text{Pat}^{(\ell)} \pi)^{-1} := \{ \sigma^{-1} \mid \sigma \in \text{Pat}^{(\ell)} \pi \},$$

$$\text{Pat}^{(\ell)} \pi \tau \subseteq (\text{Pat}^{(\ell)} \pi)(\text{Pat}^{(\ell)} \tau) = \{ \sigma \sigma' \mid \sigma \in \text{Pat}^{(\ell)} \pi, \sigma' \in \text{Pat}^{(\ell)} \tau \}.$$

Since $S$ is a group, it contains the inverses and products of its members. Consequently, $\pi^{-1}$ and $\pi \tau$ also belong to $\text{Comp}^{(n)} S$. Thus $\text{Comp}^{(n)} S$ is a group.
The converse of the Proposition does not hold.

There even exist subgroups $H \leq S_n$ which are of the form $\text{Comp}^{(n)} S$ for some $S \subseteq S_{\ell}$ but there is no subgroup $G \leq S_{\ell}$ such that $H = \text{Comp}^{(n)} G$.

However, for $\ell \leq n$ and $\ell \leq 3$, it holds that for every $S \subseteq S_{\ell}$, $\text{Comp}^{(n)} S$ is a subgroup of $S_n$ if and only if $S$ is a subgroup of $S_{\ell}$. 
Starting from $G \leq S_n$, describe the sequence $G, \text{Comp}^{(n+1)} G, \text{Comp}^{(n+2)} G, \ldots$
M. D. ATKINSON, R. BEALS,
Permuting mechanisms and closed classes of permutations,
in: C. S. Calude, M. J. Dinneen (eds.), Combinatorics, Computation & Logic,

M. D. ATKINSON, R. BEALS,
Permutation involvement and groups,
Theorem (Atkinson, Beals)

If $C$ is a permutation class in which every level $C^{(n)}$ is a permutation group, then the level sequence $C^{(1)}, C^{(2)}, \ldots$ eventually coincides with one of the following families of groups:

1. the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,
2. the natural cyclic groups $\mathbb{Z}_n$,
3. the full symmetric groups $S_n$,
4. the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).
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Let $C$ be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.

1. $C^{(n)} = S_n$ for all $n \in \mathbb{N}_+$.
2. For some $M \in \mathbb{N}$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, and $C^{(n)} = D_n$ for $n > M$.
3. For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M + 1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

(i) $C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither $D_{M+2}$ nor $Z_{M+2}$, or

(ii) $C^{(M+1)}$ is a proper overgroup of $Z_{M+1}$ but is not $D_{M+1}$.
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Group classes

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Roadmap

- $S_n$, $\langle \delta_n \rangle$, trivial
- $A_n$
- $\zeta_n \in G$ and $A_n \not\in G$
- $\zeta_n \not\in G$:
  - intransitive
  - transitive:
    - imprimitive
    - primitive

\[
\delta_n = n(n-1) \ldots 1
\]

\[
\zeta_n = (1 \ 2 \ \cdots \ n)
\]
Proposition

Let $G \leq S_n$. Then, unless $\zeta_n \notin G$ and $G$ is intransitive or imprimitive, the smallest $i \in \mathbb{N}_+$ for which $\text{Comp}^{(n+i)} G$ is one of the asymptotic groups is at most 2.

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Let \( n, m \in \mathbb{N}_+ \) with \( n \leq m \). Let \( G \leq S_n \). Then \( \delta_m \in \text{Comp}^{(m)} G \) if and only if \( \delta_n \in G \).

**Lemma**

Let \( G \leq S_n \).

(a) The following statements are equivalent.

(i) \( Z_n \leq G \).

(ii) \( Z_{n+1} \leq \text{Comp}^{(n+1)} G \).

(iii) \( \text{Comp}^{(n+1)} G \) contains a permutation \( \pi \in Z_{n+1} \setminus \{\iota_{n+1}\} \).

(b) The following statements are equivalent.

(i) \( D_n \leq G \).

(ii) \( D_{n+1} \leq \text{Comp}^{(n+1)} G \).

(iii) \( \text{Comp}^{(n+1)} G \) contains a permutation \( \pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\}) \).
The following statements hold for all $n \in \mathbb{N}_+$.

(a) \( \text{Comp}^{(n+1)} S_n = S_{n+1} \).

(b) If $n \geq 2$, then \( \text{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\} \).

(c) If $n \geq 3$, then \( \text{Comp}^{(n+1)} \langle \delta_n \rangle = \langle \delta_{n+1} \rangle \).
Let $\Pi$ be a partition of $[n]$.

\[ S_{\Pi} := \{ \pi \in S_n \mid \forall B \in \Pi : \pi(B) = B \} \]
Alternating groups

$\mathcal{OE}_{n+1}$ – partition of $[n+1]$ into odd and even numbers

$S_{\mathcal{OE}_{n+1}}$ – permutations preserving blocks of $\mathcal{OE}_{n+1}$

$W_{\mathcal{OE}_{n+1}}$ – permutations interchanging blocks of $\mathcal{OE}_{n+1}$

$A_{n+1}$ – even permutations

$O_{n+1}$ – odd permutations

Theorem

$$\text{Comp}^{(n+1)} A_n = (S_{\mathcal{OE}_{n+1}} \cap A_{n+1}) \cup (W_{\mathcal{OE}_{n+1}} \cap O_{n+1}).$$

Theorem

$$\text{Comp}^{(n+2)} A_n = \begin{cases} 
\langle \delta_{n+2} \rangle, & \text{if } n \equiv 0 \pmod{4}, \\
Z_{n+2}, & \text{if } n \equiv 1 \pmod{4}, \\
\{ \iota_{n+2} \}, & \text{if } n \equiv 2 \pmod{4}, \\
D_{n+2}, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}$$
Groups containing the natural cycle

Theorem

Let $G \leq S_n$, and assume that $G$ contains the natural cycle $\zeta_n$.

(i) If $D_n \leq G$ and $G \not\in \{S_n, A_n\}$, then $\text{Comp}^{(n+1)} G = D_{n+1}$.

(ii) If $D_n \not\leq G$, then $\text{Comp}^{(n+1)} G = Z_{n+1}$. 
Let $G \leq S_n$ be an intransitive group.

Let $\text{Orb } G$ be the set of orbits of $G$.

Then $G \leq S_{\text{Orb } G}$.

Moreover, $\text{Orb } G$ is the finest partition $\Pi$ such that $G \leq S_{\Pi}$. 
Intransitive groups

Let \( \Pi \) be a partition of \([n]\).

Define the partition \( \Pi' \) of \([n + 1]\) as follows.

Let \( I_\Pi \) be the coarsest interval partition that refines \( \Pi \).

For each \([a, b] \in I_\Pi\), we let \( \{a\} \) and \([a + 1, b]\) be blocks of \( \Pi' \).

Exceptions:
If \( a = 1 \) and \( b \neq n \), then \([a, b]\) is a block of \( \Pi' \).
If \( a \neq 1 \) and \( b = n \), then \( \{a\} \) and \([a + 1, n + 1]\) are blocks of \( \Pi' \).
If \( a = 1 \) and \( b = n \), then \([1, n + 1]\) is a block of \( \Pi' \).

Example

\[
\Pi = \{\{1, 2, 3, 7, 8, 9, 10\}, \\
\{4, 5, 6, 12, 13, 14\}, \\
\{11\}\}
\]

\[
I_\Pi = \{\{1, 2, 3\}, \\
\{4, 5, 6\}, \\
\{7, 8, 9, 10\}, \\
\{11\}, \\
\{12, 13, 14\}\}
\]

\[
\Pi' = \{\{1, 2, 3\}, \\
\{4\}, \{5, 6\}, \\
\{7\}, \{8, 9, 10\}, \\
\{11\}, \\
\{12\}, \{13, 14, 15\}\}
\]
Intransitive groups

**Theorem**

Let $\Pi$ be a partition of $[n]$.

(i) \( \text{If } \delta_n \notin S_\Pi, \text{ then } \text{Comp}^{(n+1)} S_\Pi = S_{\Pi'} \).  

(ii) \( \text{If } \delta_n \in S_\Pi, \text{ then } \text{Comp}^{(n+1)} S_\Pi = S_{\Pi'} \cup \delta_{n+1} S_{\Pi'} = \langle S_{\Pi'}, \delta_{n+1} \rangle; \text{ moreover, } \Pi' = \delta_{n+1}(\Pi') \).

**Theorem**

Let $\Pi$ be a partition of $[n]$.

(i) $\Pi'$ is an interval partition with no consecutive non-trivial blocks.

(ii) \( \text{If } \Pi \text{ is an interval partition with no consecutive non-trivial blocks and } \Pi = \delta_n(\Pi), \text{ then } \text{Comp}^{(n+1)} \langle S_\Pi, \delta_n \rangle = \langle S_{\Pi'}, \delta_{n+1} \rangle; \text{ moreover, } \Pi' = \delta_{n+1}(\Pi') \).
Intransitive groups

Theorem

Let $G \leq S_n$ be an intransitive group, and let $\Pi := \text{Orb } G$. Let $a$ and $b$ be the largest numbers $\alpha$ and $\beta$, respectively, such that $S_{\alpha,\beta}^n \leq G$. Then for all $\ell \geq M_{a,b}(\Pi)$, it holds that $\text{Comp}^{(n+\ell)} G = S_{n+\ell}^{a,b}$ or $\text{Comp}^{(n+\ell)} G = \langle S_{n+\ell}^{a,b}, \delta_{n+\ell} \rangle$.

$M(\Pi) := \max(\{|B| : B \in l^-_\Pi \} \cup \{1\})$

$M_{a,b}(\Pi) := \max(M(\Pi), |1/l_\Pi| - a + 1, |n/l_\Pi| - b + 1)$
Let $\Pi$ be a partition of $[n]$.

\[ \text{Aut } \Pi := \{ \pi \in S_n \mid \forall B \in \Pi : \pi(B) \in \Pi \} \]
Imprimitive groups

**Theorem**

Let $\Pi$ be a partition of $[n]$ with no trivial blocks. Then

$$\text{Comp}^{(n+1)} \text{Aut} \Pi = \begin{cases} 
\langle S_{\Pi'}, E_{\Pi} \rangle, & \text{if } \delta_n \notin \text{Aut} \Pi, \\
\langle S_{\Pi'}, E_{\Pi}, \delta_{n+1} \rangle, & \text{if } \delta_n \in \text{Aut} \Pi,
\end{cases}$$

where $E_{\Pi}$ is the following set of permutations:

- If $[1, \ell] \propto \Pi$ for some $\ell$ with $1 < \ell < n$, then $\nu_{\ell}^{(n+1)} \in E_{\Pi}$.
- If $[m, n] \propto \Pi$ for some $m$ with $1 < m < n$, then $\lambda_{n-m+1}^{(n+1)} \in E_{\Pi}$.
- If $[1, n] \propto \Pi$, then $\zeta_{n+1} \in E_{\Pi}$.
- $E_{\Pi}$ does not contain any other elements.
Primitive groups

**Theorem**

Assume that $G \leq S_n$ is a primitive group such that $\zeta_n \not\in G$ and $A_n \not\in G$.

(i) (a) \( n = 6 \)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{Comp}^{(n+1)} G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle (1 2 3 4), (3 4 5 6) \rangle$</td>
<td>{1234567, 2154376, 6734512, 7654321}</td>
</tr>
<tr>
<td>$\langle (1 2 3 4), (2 3 4 5 6) \rangle$</td>
<td>{1234567, 1276543, 1543276, 1567234}</td>
</tr>
<tr>
<td>$\langle (1 2 3 4 5), (3 4 5 6) \rangle$</td>
<td>{1234567, 2165437, 4561237, 5432167}</td>
</tr>
<tr>
<td>$\langle (1 2 3 4 5), (1 3 4)(2 5 6) \rangle$</td>
<td>$\langle \nu_5^{(7)} \rangle$</td>
</tr>
<tr>
<td>$\langle (2 3 4 5 6), (1 2 5)(3 4 6) \rangle$</td>
<td>$\langle \lambda_5^{(7)} \rangle$</td>
</tr>
</tbody>
</table>

(b) \( n \neq 6 \)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{Comp}^{(n+1)} G$</th>
<th>$G$</th>
<th>$\text{Comp}^{(n+1)} G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{[1,n-1]} \leq G$</td>
<td>$\langle \nu_{n-1}^{(n+1)} \rangle$</td>
<td>$D_{[2,n]} \leq G$</td>
<td>$\langle \lambda_{n-1}^{(n+1)} \rangle$</td>
</tr>
<tr>
<td>$D_{[1,n-2]} \leq G$</td>
<td>$\langle \nu_{n-2}^{(n+1)} \rangle$</td>
<td>$D_{[3,n]} \leq G$</td>
<td>$\langle \lambda_{n-2}^{(n+1)} \rangle$</td>
</tr>
</tbody>
</table>

(c) Otherwise $\text{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle$.

(ii) $\text{Comp}^{(n+2)} G \leq \langle \delta_{n+2} \rangle$. 
Thank you!