Permutation classes closed under pattern involvement and composition

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joint work with
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Permutations

We consider permutations of $\{1, \dots, n\}$, for some $n \in \mathbb{N}_+$.

On the one hand, a **permutation** is a bijective map on $\{1, \dots, n\}$.

On the other hand, a **permutation** $\pi \in S_n$ will be considered as a string of length n:

$$\pi = \pi_1 \pi_2 \dots \pi_n,$$

where $\pi_i = \pi(i)$.

Permutation patterns

$$\sigma = \sigma_1 \dots \sigma_\ell \in S_\ell \qquad \tau = \tau_1 \dots \tau_\ell \in S_n \qquad (\ell \leq n)$$

σ is a **pattern** of τ (or τ **involves** σ), in symbols, $\sigma \leq \tau$, if there exists a substring $\tau_{i_1} \dots \tau_{i_\ell}$ ($i_1 < i_2 < \cdots < i_\ell$) that is order-isomorphic to $\sigma_1 \dots \sigma_\ell$.

Example

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5 26 4
3 14 2

$3142 \leq 5132674$

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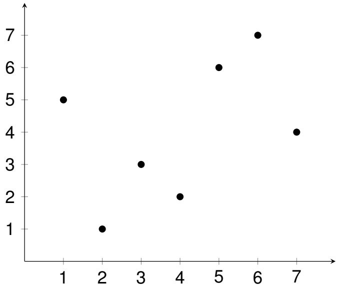

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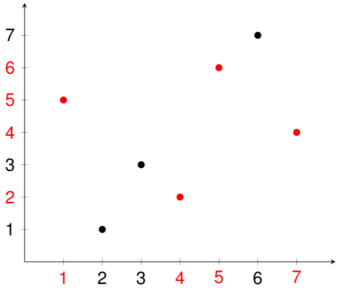

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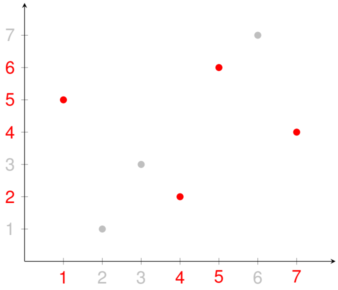

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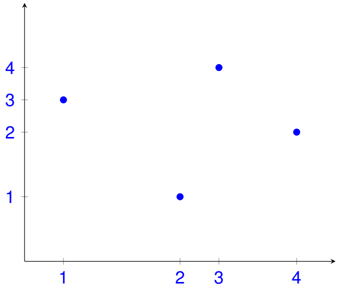

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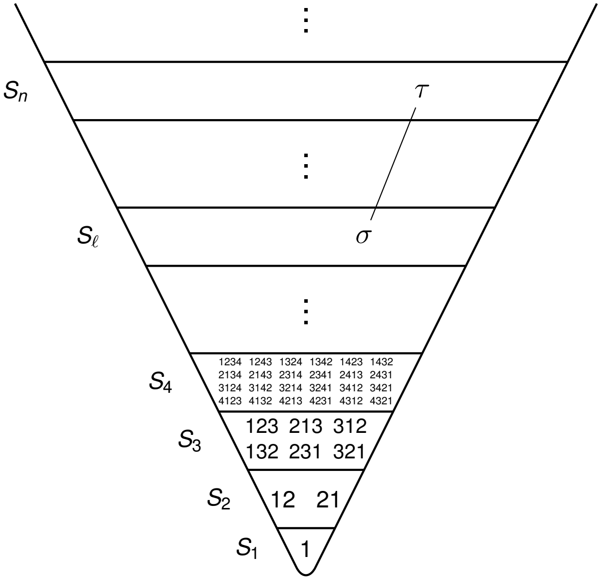
τ **avoids** σ if $\sigma \nleq \tau$.

The pattern involvement relation $\leq$ is a partial order on the set $\mathbb{P} := \bigcup_{n \geq 1} S_n$ of all finite permutations. Downward closed subsets of $\mathbb{P}$ under $\leq$ are called **permutation classes**.

Permutation patterns



$\sigma \leq \tau$

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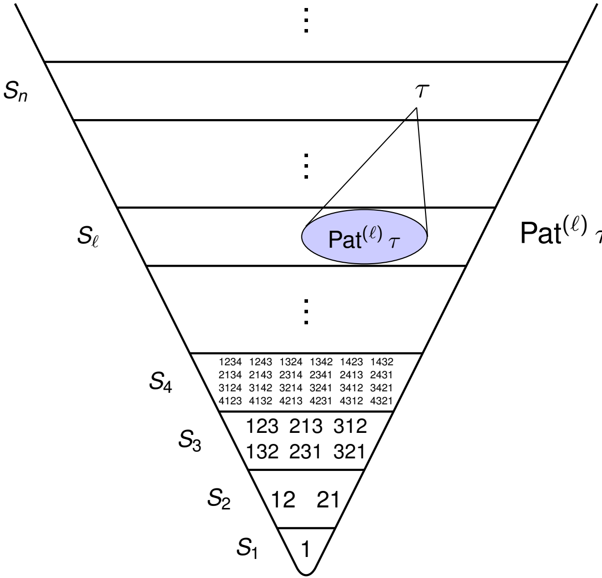

$$\mathrm{Pat}^{(\ell)}\,\tau = \{\sigma \in S_\ell \mid \sigma \leq \tau\}$$

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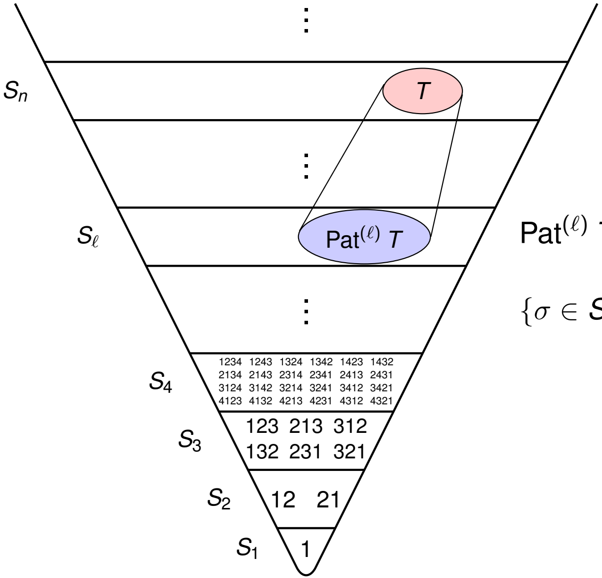
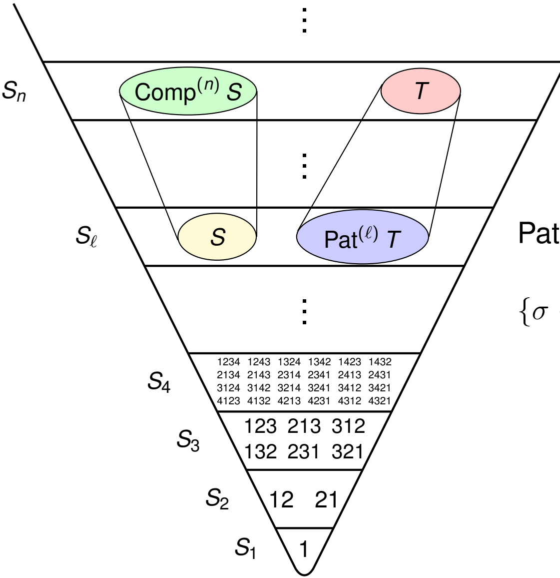


$$\mathrm{Pat}^{(\ell)}\, T = \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\, \tau = \{\sigma \in S_\ell \mid \exists \tau \in T\colon \sigma \leq \tau\}$$



$$\mathrm{Comp}^{(n)}\, S = \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)}\, \tau \subseteq S\}$$

$$\mathrm{Pat}^{(\ell)}\, T = \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\, \tau = \{\sigma \in S_\ell \mid \exists \tau \in T \colon \sigma \leq \tau\}$$

Galois connection

The operators $\mathrm{Pat}^{(\ell)}$ and $\mathrm{Comp}^{(n)}$ constitute a monotone Galois connection between $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$.

This is in fact the monotone Galois connection induced by the pattern avoidance relation $\nleq$ between S_ℓ and S_n.

$S \subseteq S_\ell$, $T \subseteq S_n$ ($\ell \leq n$)

$$\mathrm{Comp}^{(n)}\, S := \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)}\, \tau \subseteq S\} = \{\tau \in S_n \mid \forall \sigma \in S_\ell \setminus S \colon \sigma \nleq \tau\},$$

$$\mathrm{Pat}^{(\ell)}\, T := \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\, \tau = S_\ell \setminus \{\sigma \in S_\ell \mid \forall \tau \in T \colon \sigma \nleq \tau\}.$$

Closures and kernels:

$$\mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, S \subseteq S,$$

$$T \subseteq \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, T,$$

$$\mathrm{Comp}^{(n)}\, S = \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, S,$$

$$\mathrm{Pat}^{(\ell)}\, T = \mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, T.$$

Galois connection

Further details on this Galois connection and the related Galois connection between the subgroup lattices of S_ℓ and S_n in

E. Lehtonen, R. Pöschel,
Permutation groups, pattern involvement, and Galois connections, arXiv:1605.04516.

The group case for $\mathrm{Comp}^{(n)} S$

Proposition

If S is a subgroup of S_ℓ, then $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n.

Sketch of a proof.

Assume that $S \leq S_\ell$. Let $\pi, \tau \in \mathrm{Comp}^{(n)} S$.
Thus $\mathrm{Pat}^{(\ell)} \pi, \mathrm{Pat}^{(\ell)} \tau \subseteq S$.
It holds that

$$\mathrm{Pat}^{(\ell)} \pi^{-1} = (\mathrm{Pat}^{(\ell)} \pi)^{-1} := \{\sigma^{-1} \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi\},$$

$$\mathrm{Pat}^{(\ell)} \pi\tau \subseteq (\mathrm{Pat}^{(\ell)} \pi)(\mathrm{Pat}^{(\ell)} \tau) = \{\sigma\sigma' \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi,\ \sigma' \in \mathrm{Pat}^{(\ell)} \tau\}.$$

Since S is a group, it contains the inverses and products of its members. Consequently, π^{-1} and $\pi\tau$ also belong to $\mathrm{Comp}^{(n)} S$. Thus $\mathrm{Comp}^{(n)} S$ is a group. □

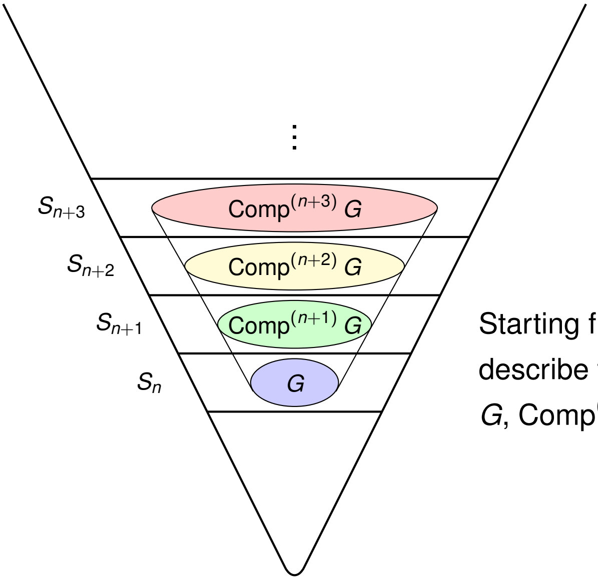
The group case for $\mathrm{Comp}^{(n)} S$, continued

The converse of the Proposition does not hold.

There even exist subgroups $H \leq S_n$ which are of the form $\mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$ but there is no subgroup $G \leq S_\ell$ such that $H = \mathrm{Comp}^{(n)} G$.

However, for $\ell \leq n$ and $\ell \leq 3$, it holds that for every $S \subseteq S_\ell$, $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n if and only if S is a subgroup of S_ℓ.

Permutation groups arising from pattern avoidance



Starting from $G \leq S_n$, describe the sequence $G, \mathrm{Comp}^{(n+1)}\, G, \mathrm{Comp}^{(n+2)}\, G, \ldots$

Group classes

M. D. ATKINSON, R. BEALS,
Permuting mechanisms and closed classes of permutations,
in: C. S. Calude, M. J. Dinneen (eds.), *Combinatorics, Computation & Logic,* Proc. DMTCS '99 and CATS '99 (Auckland), Aust. Comput. Sci. Commun., 21, No. 3, Springer, Singapore, 1999, pp. 117–127.

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Permutation involvement and groups,
Q. J. Math. **52** (2001) 415–421.

Group classes

Theorem (Atkinson, Beals)

If C is a permutation class in which every level $C^{(n)}$ is a permutation group, then the level sequence $C^{(1)}, C^{(2)}, \dots$ eventually coincides with one of the following families of groups:

(1) *the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,*

(2) *the natural cyclic groups Z_n,*

(3) *the full symmetric groups S_n,*

(4) *the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).*

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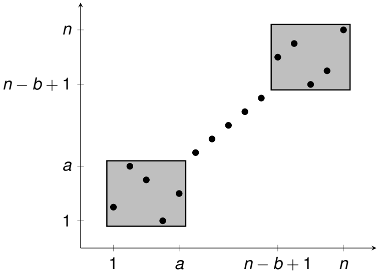
$S_n^{a,b}$



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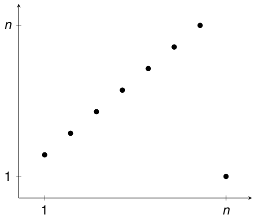
$$\zeta_n = (1\ 2\ \cdots n)$$



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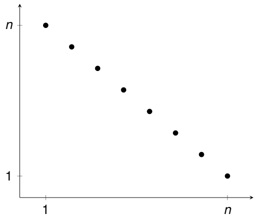
$$\delta_n = n(n-1)\dots 1$$



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Theorem (Atkinson, Beals)

Let C be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.

(1) $C^{(n)} = S_n$ for all $n \in \mathbb{N}_+$.

(2) For some $M \in \mathbb{N}$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, and $C^{(n)} = D_n$ for $n > M$.

(3) For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M + 1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

(i) $C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither D_{M+2} nor Z_{M+2}, or

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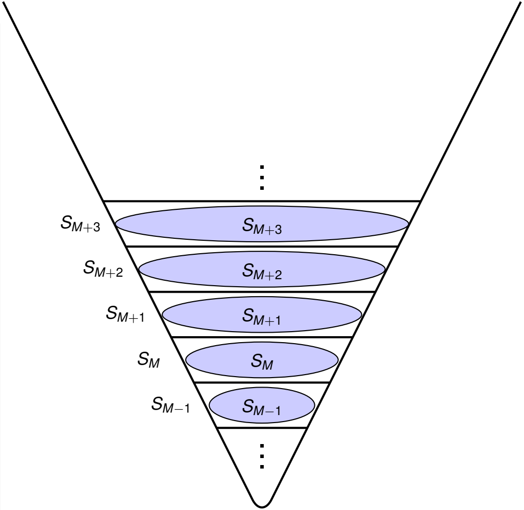

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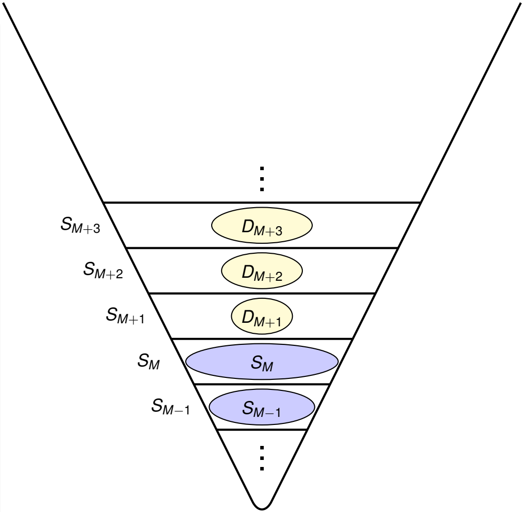

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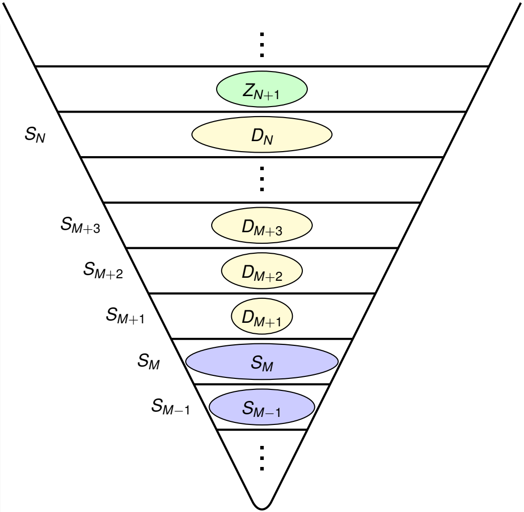

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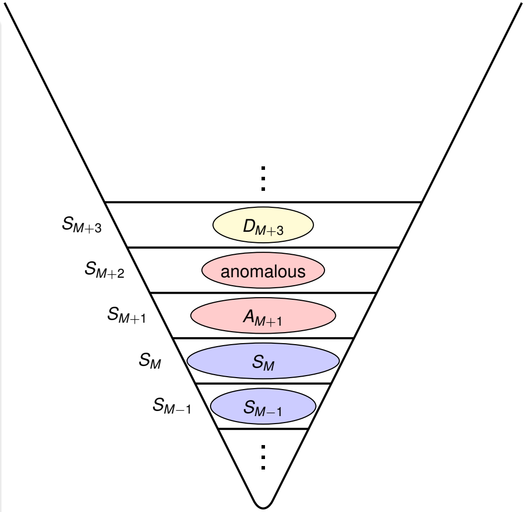

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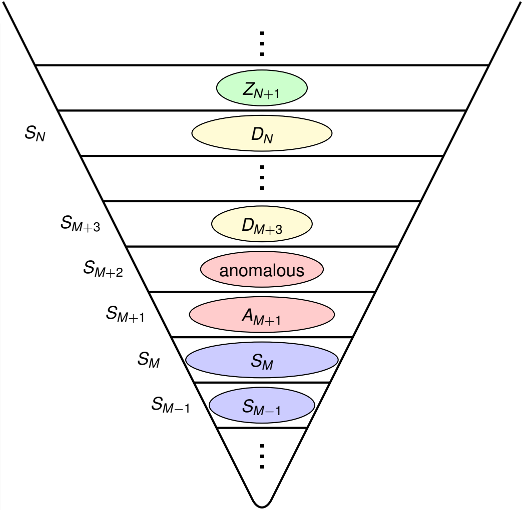

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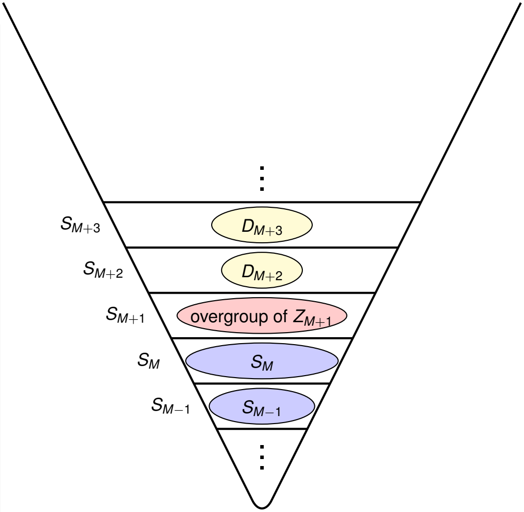

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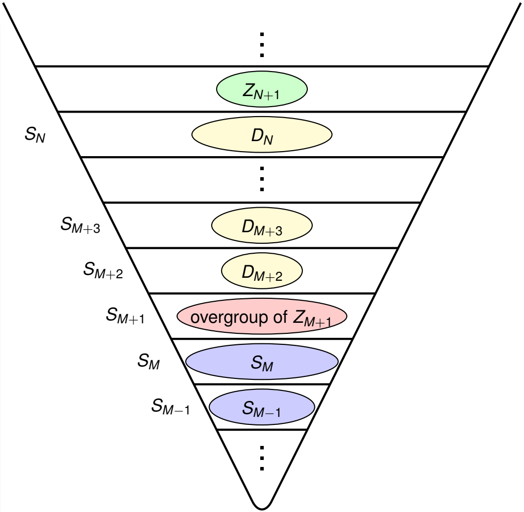
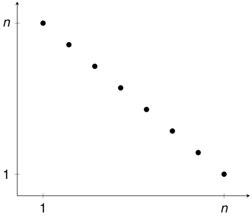


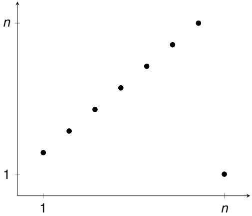
Roadmap

- S_n, $\langle \delta_n \rangle$, trivial
- A_n
- $\zeta_n \in G$ and $A_n \not\leq G$
- $\zeta_n \notin G$:
  - intransitive
  - transitive:
    - imprimitive
    - primitive

$$\delta_n = n(n-1)\dots 1$$



$$\zeta_n = (1\ 2\ \cdots n)$$



Executive summary

Proposition

Let $G \leq S_n$. Then, unless $\zeta_n \notin G$ and G is intransitive or imprimitive, the smallest $i \in \mathbb{N}_+$ for which $\mathrm{Comp}^{(n+i)} G$ is one of the asymptotic groups is at most 2.

E. LEHTONEN,
Permutation groups arising from pattern involvement,
arXiv:1605.05571.

Simple observations

Lemma

Let $n, m \in \mathbb{N}_+$ *with* $n \leq m$*. Let* $G \leq S_n$*. Then* $\delta_m \in \mathrm{Comp}^{(m)} G$ *if and only if* $\delta_n \in G$*.*

Lemma

Let $G \leq S_n$*.*

(a) *The following statements are equivalent.*
  (i) $Z_n \leq G$.
  (ii) $Z_{n+1} \leq \mathrm{Comp}^{(n+1)} G$.
  (iii) $\mathrm{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$.

(b) *The following statements are equivalent.*
  (i) $D_n \leq G$.
  (ii) $D_{n+1} \leq \mathrm{Comp}^{(n+1)} G$.
  (iii) $\mathrm{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$.

Symmetric, trivial, . . .

Theorem

The following statements hold for all $n \in \mathbb{N}_+$.

(a) $\mathrm{Comp}^{(n+1)} S_n = S_{n+1}$.

(b) *If $n \geq 2$, then $\mathrm{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\}$.*

(c) *If $n \geq 3$, then $\mathrm{Comp}^{(n+1)} \langle \delta_n \rangle = \langle \delta_{n+1} \rangle$.*

Notation

Let Π be a partition of $[n]$.

$$S_\Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) = B\}$$

Alternating groups

Œ_{n+1} – partition of $[n+1]$ into odd and even numbers
$S_{\text{Œ}_{n+1}}$ – permutations preserving blocks of Œ_{n+1}
$W_{\text{Œ}_{n+1}}$ – permutations interchanging blocks of Œ_{n+1}
A_{n+1} – even permutations
O_{n+1} – odd permutations

Theorem

$$\operatorname{Comp}^{(n+1)} A_n = (S_{\text{Œ}_{n+1}} \cap A_{n+1}) \cup (W_{\text{Œ}_{n+1}} \cap O_{n+1}).$$

Theorem

$$\operatorname{Comp}^{(n+2)} A_n = \begin{cases} \langle \delta_{n+2} \rangle, & \text{if } n \equiv 0 \pmod 4, \\ Z_{n+2}, & \text{if } n \equiv 1 \pmod 4, \\ \{\iota_{n+2}\}, & \text{if } n \equiv 2 \pmod 4, \\ D_{n+2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Groups containing the natural cycle

Theorem

Let $G \leq S_n$, and assume that G contains the natural cycle ζ_n.

(i) *If $D_n \leq G$ and $G \notin \{S_n, A_n\}$, then $\operatorname{Comp}^{(n+1)} G = D_{n+1}$.*

(ii) *If $D_n \nleq G$, then $\operatorname{Comp}^{(n+1)} G = Z_{n+1}$.*

Intransitive groups

Let $G \leq S_n$ be an intransitive group.

Let $\operatorname{Orb} G$ be the set of orbits of G.

Then $G \leq S_{\operatorname{Orb} G}$.

Moreover, $\operatorname{Orb} G$ is the finest partition Π such that $G \leq S_\Pi$.

Intransitive groups

Let Π be a partition of $[n]$.

Define the partition Π' of $[n+1]$ as follows.

Let I_Π be the coarsest interval partition that refines Π.

For each $[a, b] \in I_\Pi$, we let $\{a\}$ and $[a+1, b]$ be blocks of Π'.

Exceptions:

If $a = 1$ and $b \neq n$, then $[a, b]$ is a block of Π'.

If $a \neq 1$ and $b = n$, then $\{a\}$ and $[a+1, n+1]$ are blocks of Π'.

If $a = 1$ and $b = n$, then $[1, n+1]$ is a block of Π'.

Example

$$\Pi = \{\{1,2,3,7,8,9,10\}, \{4,5,6,12,13,14\}, \{11\}\}$$

$$I_\Pi = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9,10\}, \{11\}, \{12,13,14\}\}$$

$$\Pi' = \{\{1,2,3\}, \{4\}, \{5,6\}, \{7\}, \{8,9,10\}, \{11\}, \{12\}, \{13,14,15\}\}$$

Intransitive groups

Theorem

Let Π *be a partition of* $[n]$*.*

(i) *If* $\delta_n \notin S_\Pi$*, then* $\operatorname{Comp}^{(n+1)} S_\Pi = S_{\Pi'}$*.*

(ii) *If* $\delta_n \in S_\Pi$*, then* $\operatorname{Comp}^{(n+1)} S_\Pi = S_{\Pi'} \cup \delta_{n+1} S_{\Pi'} = \langle S_{\Pi'}, \delta_{n+1} \rangle$*; moreover,* $\Pi' = \delta_{n+1}(\Pi')$*.*

Theorem

Let Π *be a partition of* $[n]$*.*

(i) Π' *is an interval partition with no consecutive non-trivial blocks.*

(ii) *If* Π *is an interval partition with no consecutive non-trivial blocks and* $\Pi = \delta_n(\Pi)$*, then* $\operatorname{Comp}^{(n+1)} \langle S_\Pi, \delta_n \rangle = \langle S_{\Pi'}, \delta_{n+1} \rangle$*; moreover,* $\Pi' = \delta_{n+1}(\Pi')$*.*

Intransitive groups

Theorem

Let $G \leq S_n$ be an intransitive group, and let $\Pi := \operatorname{Orb} G$. Let a and b be the largest numbers α and β, respectively, such that $S_n^{\alpha,\beta} \leq G$. Then for all $\ell \geq M_{a,b}(\Pi)$, it holds that $\operatorname{Comp}^{(n+\ell)} G = S_{n+\ell}^{a,b}$ or $\operatorname{Comp}^{(n+\ell)} G = \langle S_{n+\ell}^{a,b}, \delta_{n+\ell} \rangle$.

$$M(\Pi) := \max(\{|B| : B \in I_\Pi^-\} \cup \{1\})$$
$$M_{a,b}(\Pi) := \max(M(\Pi), |1/I_\Pi| - a + 1, |n/I_\Pi| - b + 1)$$

Notation

Let Π be a partition of $[n]$.

$$\operatorname{Aut}\Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) \in \Pi\}$$

Imprimitive groups

Theorem

Let Π *be a partition of* $[n]$ *with no trivial blocks. Then*

$$\operatorname{Comp}^{(n+1)} \operatorname{Aut} \Pi = \begin{cases} \langle S_{\Pi'}, E_\Pi \rangle, & \text{if } \delta_n \notin \operatorname{Aut} \Pi, \\ \langle S_{\Pi'}, E_\Pi, \delta_{n+1} \rangle, & \text{if } \delta_n \in \operatorname{Aut} \Pi, \end{cases}$$

where E_Π *is the following set of permutations:*

- *If* $[1, \ell] \propto \Pi$ *for some* ℓ *with* $1 < \ell < n$*, then* $\nu_\ell^{(n+1)} \in E_\Pi$*.*
- *If* $[m, n] \propto \Pi$ *for some* m *with* $1 < m < n$*, then* $\lambda_{n-m+1}^{(n+1)} \in E_\Pi$*.*
- *If* $[1, n] \propto \Pi$*, then* $\zeta_{n+1} \in E_\Pi$*.*
- E_Π *does not contain any other elements.*

Primitive groups

Theorem

Assume that $G \leq S_n$ is a primitive group such that $\zeta_n \notin G$ and $A_n \nleq G$.

(i) (a) $n = 6$

G	$\mathrm{Comp}^{(n+1)} G$
$\langle (1\ 2\ 3\ 4), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2154376, 6734512, 7654321\}$
$\langle (1\ 2\ 3\ 4), (2\ 3\ 4\ 5\ 6) \rangle$	$\{1234567, 1276543, 1543276, 1567234\}$
$\langle (1\ 2\ 3\ 4\ 5), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2165437, 4561237, 5432167\}$
$\langle (1\ 2\ 3\ 4\ 5), (1\ 3\ 4)(2\ 5\ 6) \rangle$	$\langle \nu_5^{(7)} \rangle$
$\langle (2\ 3\ 4\ 5\ 6), (1\ 2\ 5)(3\ 4\ 6) \rangle$	$\langle \lambda_5^{(7)} \rangle$

(b) $n \neq 6$

G	$\mathrm{Comp}^{(n+1)} G$	G	$\mathrm{Comp}^{(n+1)} G$
$D_{[1,n-1]} \leq G$	$\langle \nu_{n-1}^{(n+1)} \rangle$	$D_{[2,n]} \leq G$	$\langle \lambda_{n-1}^{(n+1)} \rangle$
$D_{[1,n-2]} \leq G$	$\langle \nu_{n-2}^{(n+1)} \rangle$	$D_{[3,n]} \leq G$	$\langle \lambda_{n-2}^{(n+1)} \rangle$

(c) *Otherwise* $\mathrm{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle$.

(ii) $\mathrm{Comp}^{(n+2)} G \leq \langle \delta_{n+2} \rangle$.

The end

Thank you!