# Regular subsemigroups of the semigroup of fence-preserving partial transformations on $\mathbb{N}$

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$\preceq$ binary relation on $X$

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  reflexive : $\forall a \in X,\; a \preceq a$

  antisymmetric : $\forall a, b \in X\, (a \preceq b \;\&\; b \preceq a) \Rightarrow a = b$

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$(X, \preceq)$ is called a linearly ordered set or chain

# Transformation

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- A partial transformation $\alpha : Y \to X$ is order-preserving

$$\forall x, y \in Y,\ x \preceq y \Rightarrow x\alpha \preceq y\alpha$$

  $OP(X) :=$ the set of all order-preserving partial transformations on $X$

# Transformation

- In 2004, A. Laradji and A. Umar, Combinatorial results for semigroups of order-preserving partial transformations
- In 2010, W. Mora and Y. Kemprasit, Regular elements of some order-preserving transformation semigroups
- In 2011, I. Dimitrova and J. Koppitz, On the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations
- In 2015, P. Zhao, H. Hu and T. You, Maximal regular subsemibands of the finite order-preserving partial transformation semigroups $PO(n,r)$

# Fence

A fence X, also called zigzag poset, is a partially ordered set $(X, \preceq)$ in which the order relation forms a path with alternating orientations:

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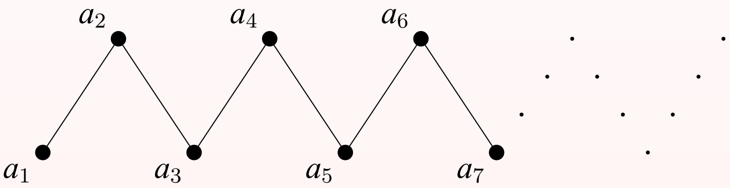


up-fence

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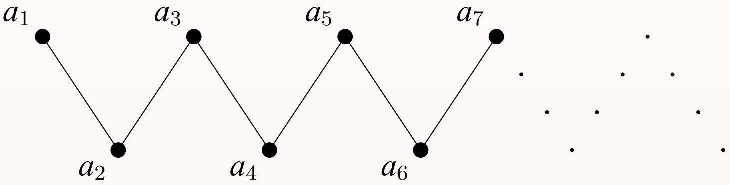


down-fence

where $X = \{a_1, a_2, a_3, \ldots\}$.

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The number of antichains in a fence is Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

# fence-preserving partial transformations on $\mathbb{N}$

**Definition**

*Let $\mathbb{N}$ be the set of natural numbers. We define a partial order $\preceq$ on $\mathbb{N}$ by*

$$n \prec n+1 \quad \text{if} \quad n \text{ is odd}$$
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## Example

$\alpha : \mathbb{N} \setminus \{1, 2\} \to \mathbb{N}$ *defined by* $i\alpha = i - 2$ *for all* $i \in \mathbb{N} \setminus \{1, 2\}$ *is a fecne-preserving partial transformation on* $\mathbb{N}$.

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- $Reg(FP(\mathbb{N}))$ := the set of all fence-preserving partial transformations on $\mathbb{N}$ which is regular in $FP(\mathbb{N})$
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  $:= \{\alpha \in FP(\mathbb{N}) : \alpha \text{ is regular in } FP(\mathbb{N})\}$
- $id_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ defined by $x id_{\mathbb{N}} = x$ for all $x \in \mathbb{N}$

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- $A \nparallel B$ : for each $x \in A, x \nparallel y$ for all $y \in B$

# Results

**Theorem**

*Let* $\alpha \in FP(\mathbb{N})$. *Then* $\alpha$ *is regular if and only if there exists a subset* $Y \subseteq$ *dom*$\alpha$ *such that* $\alpha|_Y$ *is a bijection from* $Y$ *to ran*$\alpha$ *and* $(\alpha|_Y)^{-1} \in FP(\mathbb{N})$.

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**Corollary**

*Let $\alpha \in FP(\mathbb{N})$ with $\mid ran\alpha \mid = 2$. Then $\alpha$ is regular if and only if $ran\alpha$ is not a fence or there is a two-element subfence $Y$ of $dom\alpha$ such that $Y\alpha = ran\alpha$.*

$A := \{\alpha \in Reg(FP(\mathbb{N})) : | ran\alpha | \leq 2\}$

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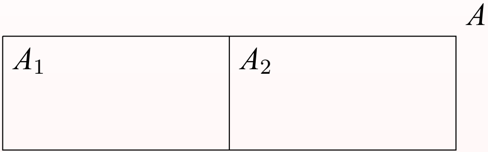


$$A_1 := \{\alpha \in A : ran\alpha \text{ is a fence}\}$$

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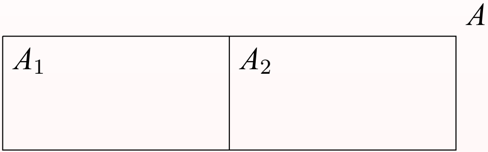


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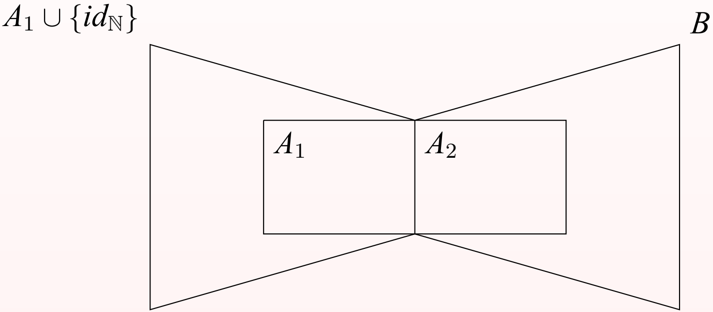
- $A_2 \subseteq B$



# Maximal regular subsemigroups of $FP(\mathbb{N})$

**Theorem**

$A_1 \cup \{id_{\mathbb{N}}\} = \{\alpha \in Reg(FP(\mathbb{N})) : ran\alpha \text{ is a fence and } \mid ran\alpha \mid \leq 2\}$ $\cup\{id_{\mathbb{N}}\}$ *is a maximal regular subsemigroup of* $FP(\mathbb{N})$.

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# Maximal regular subsemigroups of $FP(\mathbb{N})$

Let $a \in \mathbb{N} \setminus \{1, 2, 3\}$ and let $C_a$ be the set of all $\alpha \in Reg(FP(\mathbb{N}))$ with

(i) $a \notin ran\alpha$ or $ran\alpha \subseteq \{a-1, a, a+1\}$

(ii) $a \notin dom\alpha$ or $| a\alpha\alpha^{-1} | \geq 2$ or $ran\alpha = ran\big(\alpha|_{\{a-1,a,a+1\}}\big)$

(iii) $\alpha|_{dom\alpha\setminus\{a\}} \in B$.

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**Theorem**

*$C_a^* := C_a \cup \{id_{\mathbb{N}}\}$ is a maximal regular subsemigroup of $FP(\mathbb{N})$.*

**Corollary**

*Let $a, b \in \mathbb{N} \setminus \{1,2,3\}$ with $a \neq b$. Then $C_a^* \neq C_b^*$*

# References


- [1] I. Dimitrova and J. Koppitz, **On the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations**, Semigroup Forum **82**(1) (2011)
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Thank you for your attention