The lattice of linear Mal'cev conditions

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Identities of height at most 1 are usually called linear.

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Not examples

group terms, lattice terms, semilattice term.

Definition (Barto, Pinsker)

An algebra **A** is said to be a reflection of **B** defined by mappings $h_1 \colon B \to A$ and $h_2 \colon A \to B$, if for every basic operation f we have

$$f_{\mathbf{A}}(a_1, \ldots, a_n) = af_{\mathbf{B}}(b(a_1), \ldots, b(a_n)).$$

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For a clone $\mathcal{B}$, **A** is a retraction (reflection, resp.) of $\mathcal{B}$ if **A** is a retraction (reflection, resp.) of the algebra $(B, (f)_{f \in \mathcal{B}})$.

Linear Birkhoff

The class of all retractions (reflections, resp.) of algebras from $\mathcal{K}$ is denoted $\mathbf{R}\,\mathcal{K}$ ($\mathbf{R}_{\text{ret}}\,\mathcal{K}$).

Theorem (Barto, Pinsker, O)

A class of algebras is definable by linear identities (identities of height 1, resp.) if and only if it is closed under $\mathbf{R}_{ret}$ and $\mathbf{P}$ ($\mathbf{R}$ and $\mathbf{P}$, resp.).

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(Except congruence singularity.)

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Meet of two Mal'cev conditions is the strongest Mal'cev condition that which is weaker then both of the original ones.

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$$\begin{aligned} xy \cdot zw &\approx xw \\ f(x_0x_1, y_0y_1, z_0z_1) &\approx f(x_0, y_0, z_0) \cdot f(x_1, y_1, z_1) \text{ for } f \in \{q, d_1, \ldots, d_n\}. \end{aligned}$$

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Meet of Mal'cev and Jónsson terms (cont.)

Observation

A clone satisfies this Mal'cev condition if and only if it is a product of a clone with Mal'cev operation and a clone with Jónsson terms.

A product of clones $\mathcal{A}$ and $\mathcal{B}$ is the clone $\mathcal{C}$ with $C = A \times B$, and $\mathcal{C}^{[n]} = \{f \times g \colon f \in \mathcal{A}^{[n]}, g \in \mathcal{B}^{[n]}\}$.

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Thank you for your attention!