Poset-based logic for the De Morgan negation

Jan Paseka
Department of Mathematics and Statistics
Masaryk University
Brno, Czech Republic
paseka@math.muni.cz

Ivan Chajda
Department of Algebra and Geometry
Palacký University Olomouc
Olomouc, Czech Republic
ivan.chajda@upol.cz

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Introduction - poset-based logic for the De Morgan negations

In this lecture we introduce several poset-based logics for the De Morgan negation.

A crucial problem concerning logics is their soundness, completeness, finite model property and decidability. We solved this problem with I. Chajda for poset-based logics for the De Morgan negation.

Note that the study of this problem goes back to the ISMVL in Toyama where we started the study of this problem during our project **Algebraic methods in Quantum Logic** supervised by **Ivo G. Rosenberg**.

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1. Introduction - Poset-based logic for the De Morgan negation

2. Warming up - basic notions, definitions and results
   - The poset-based logic with the De Morgan negation
   - Representation theorem of De Morgan posets
   - Soundness and completeness

3. The normal D-modal poset-based logic with the De Morgan negation
   - Introducing the logic
   - Semi-tense De Morgan algebras
   - Soundness and completeness of $\mathbf{L}_{DmD}$

4. Finite model property and decidability for poset-based logics for the De Morgan negation

The poset-based logic with the De Morgan negation

In what follows, we denote propositional variables by $p;q;r;\ldots$, the logical connective negation by $\neg$, and two logical constants $\bot$ and $\top$ where $\bot$ stands for the contradiction and $\top$ stands for the tautology. So formulae are inductively defined by the following BNF:

$$\phi ::= p \mid \neg\phi \mid \bot \mid \top.$$

We denote formulae by $\phi, \psi, \chi, \ldots$ and let Φ and Λ be the set of all propositional variables and the set of all formulae. A logical consequence relation $\mapsto$ is a binary relation on Λ. We may interpret $\phi \mapsto \psi$ as "if ϕ then ψ." So we call the left-hand formulae premises and the right-hand formulae conclusions. We may sometimes call logical consequences sequents.

The poset-based logic with the De Morgan negation

We now introduce a sequent calculus (L_{DMP}) given as follows.

$$\frac{}{\phi \Rrightarrow \phi}\ \text{(Ax)} \qquad \frac{}{\neg\neg\phi \Rrightarrow \phi}\ \text{(DN-l)}$$

$$\frac{}{\bot \Rrightarrow \phi}\ \text{(bot)} \qquad \frac{}{\phi \Rrightarrow \top}\ \text{(top)}$$

$$\frac{\phi \Rrightarrow \neg\psi}{\psi \Rrightarrow \neg\phi}\ (\neg\text{-r}) \qquad \frac{\phi \Rrightarrow \psi \quad \psi \Rrightarrow \chi}{\phi \Rrightarrow \chi}\ \text{(cut)}$$

The poset-based logic for the De Morgan negation is the collection of all sequents derivable in L_{DMP}, denoted by $\mathbf{L}_{DMP}$.

The poset-based logic with the De Morgan negation

Using previous axioms and inductive steps, we can derive several useful rules as follows.

Proposition

In the sequent calculus L_{DMP}, we can derive the following theorems and inference rules.

$$\frac{}{\phi \Rrightarrow \neg\neg\phi} \quad \text{(DN-r)}$$

$$\frac{\phi \Rrightarrow \psi}{\neg\psi \Rrightarrow \neg\phi} \quad (\neg\text{-}\neg) \qquad \frac{\neg\phi \Rrightarrow \psi}{\neg\psi \Rrightarrow \phi} \quad (\neg\text{-l})$$

$$\frac{}{\top \Rrightarrow \neg\bot} \quad \text{(tnb)} \qquad \frac{}{\neg\top \Rrightarrow \bot} \quad \text{(ntb)}$$

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ be a bounded ordered set with an antitone involution (i.e., 0 is the least and 1 the greatest element). Then $\mathbf{A}$ is called a *De Morgan poset*. We notice that $0' = 1$, $1' = 0$ and $\mathbf{A}$ satisfies the so-called *De Morgan laws*:

$$(a \vee b)' = a' \wedge b' \quad \text{if } a \vee b \text{ exists} \quad \text{and}$$
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Let $h\colon A \to B$ be a partial mapping of De Morgan posets. We say that the partial mapping $h^{\partial}: A \to B$ is the **dual of** h if $h^{\partial}(a)$ is defined for all $a \in A$ such that $a' \in dom\, h$ in which case

$$h^{\partial}(a) = h(a')'.$$

A *morphism of De Morgan posets* is a mapping between De Morgan posets which is a morphism of the respective bounded posets and which preserves the antitone involution.

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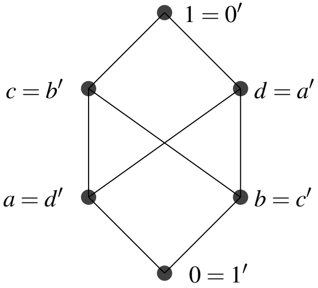
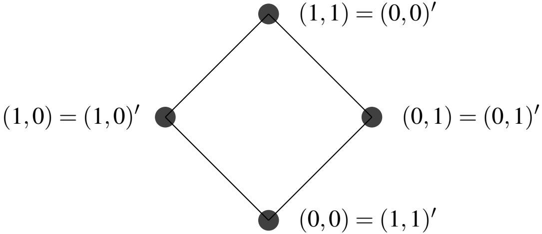


Example

The De Morgan poset $\mathbf{M} = (M; \leq, ', 0, 1)$, $M = \{0, a, b, c, d, 1\}$ displayed by the Hasse diagram in the above figure is the smallest non-lattice De Morgan poset.

A canonical example of a De Morgan poset is the four-element De Morgan poset $\mathbf{M}_2$ depicted below.



Recall that the four-element De Morgan poset $\mathbf{M}_2$, considered as a distributive De Morgan lattice, generates the variety of all distributive De Morgan lattices. This result was the motivation for our study of the representation theorem of De Morgan posets.

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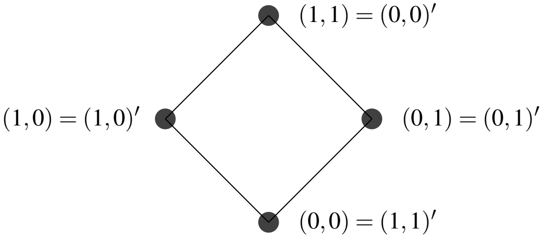

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Definition

Let $\mathbf{A}$ be a De Morgan poset and $\mathbf{M}$ be a complete De Morgan lattice. Let S be a set of morphisms of De Morgan posets from $\mathbf{A}$ to $\mathbf{M}$.

(a) S is called *order reflecting* or a *full set* if

$$((\forall s \in S)\, s(a) \leq s(b)) \implies a \leq b$$

for any elements $a, b \in A$.

(b) If S is an order-reflecting set then $\mathbf{A}$ is said to be *representable in* $\mathbf{M}$.

For any De Morgan poset $\mathbf{A} = (A; \leq, ', 0, 1)$, we let $T_{\mathbf{A}}^{\text{DMP}}$ denote a set of morphisms of De Morgan posets into the four-element De Morgan poset $\mathbf{M}_2$.

Proposition

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ be a De Morgan poset. Then the map $i_{\mathbf{A}} : A \to M_2^{T_{\mathbf{A}}^{DMP}}$ given by $i_{\mathbf{A}}(a)(s) = s(a)$ for all $a \in A$ and all $s \in T_{\mathbf{A}}^{DMP}$ is an order-reflecting morphism of De Morgan posets such that $i_{\mathbf{A}}(A)$ is a De Morgan subposet of $\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}}$.

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The key question is the soundness of the given logic, i.e. its correspondence to the given algebraic structure. Fortunately, we are able to prove the following theorem.

Theorem

(Soundness). The poset-based logic for the De Morgan negation $\mathbf{L}_{DMP}$ is sound for the class of De Morgan posets. That is, for every sequent $\phi \Rrightarrow \psi$ in $\mathbf{L}_{DMP}$, $s_\phi \leq t_\psi$ is valid on all De Morgan posets, where s_ϕ and t_ψ are the corresponding term functions for ϕ and ψ.

The second important property of a given logic is its completeness. Similarly as for the classical logic, we prove the following assertion by using of the corresponding Lindenbaum-Tarski algebra.

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Idea of the proof: We take the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP}$ for $\mathbf{L}_{DMP}$. That is, we take the quotient of Λ with respect to the equivalence relation $\equiv$, defined by $\phi \equiv \psi \iff \phi \Rrightarrow \psi$ in $\mathbf{L}_{DMP}$ and $\psi \Rrightarrow \phi$ in $\mathbf{L}_{DMP}$. It is plain that $\equiv$ is really an equivalence relation. On this quotient set $\Lambda/_{\equiv}$, we can define

- $0 := [\bot]_{\equiv}$,
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It suffices to show that the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP} = (\Lambda/_{\equiv}, \leq, ', 0, 1)$ is a De Morgan poset.

Remark

It is a plain checking that the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP}$ is a free De Morgan poset over the set of propositional variables, i.e., for any mapping e from the set Φ of propositional variables to a De Morgan poset $\mathbb{A}$ there is a uniquely determined morphism of De Morgan posets $\bar{e} : \mathbb{A}_{DMP} \to \mathbb{A}$ such that $e(p) = \bar{e}([p]_{\equiv})$ for all propositional variables p.

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It suffices to show that the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP} = (\Lambda/_{\equiv}, \leq, ', 0, 1)$ is a De Morgan poset.

Remark

It is a plain checking that the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP}$ is a free De Morgan poset over the set of propositional variables, i.e., for any mapping e from the set Φ of propositional variables to a De Morgan poset $\mathbf{A}$ there is a uniquely determined morphism of De Morgan posets $\widetilde{e} : \mathbb{A}_{DMP} \to \mathbf{A}$ such that $e(p) = \widetilde{e}([p]_{\equiv})$ for all propositional variables p.

In what follows, we will able to show a little bit more.

Idea of the proof: We take the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP}$ for $\mathbf{L}_{DMP}$. That is, we take the quotient of Λ with respect to the equivalence relation $\equiv$, defined by $\phi \equiv \psi \iff \phi \Rightarrow \psi$ in $\mathbf{L}_{DMP}$ and $\psi \Rightarrow \phi$ in $\mathbf{L}_{DMP}$. It is plain that $\equiv$ is really an equivalence relation. On this quotient set $\Lambda/_{\equiv}$, we can define

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Informally, an equation is any equation in the first-order language of De Morgan posets.

A *valuation* e on a De Morgan poset $\mathbf{A}$ is a morphism of De Morgan posets from the Lindenbaum-Tarski algebra $\mathbb{A}_{DMP}$ to the De Morgan poset $\mathbf{A}$. Note that e is uniquely determined by the action on propositional variables.

The notion of satisfaction (validity) of such an equation in a De Morgan poset $\mathbf{A}$ is the usual model-theoretic notion of satisfaction (validity).
The role of $\mathbf{M}_2$ in the equational theory of De Morgan posets is revealed by the following theorem.

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Outline

In what follows, we denote propositional variables by $p;q;r;\ldots$, the logical connective negation by $\neg$, two logical constants $\bot$ and $\top$, and two unary modal connectives $\Box$ (necessity operator) and $\diamond$ (possibility operator). So formulae are inductively defined by the following BNF:

$$\phi ::= p \mid \neg\phi \mid \Box\phi \mid \diamond\phi \mid \bot \mid \top.$$

We denote formulae by $\phi, \psi, \chi, \ldots$ and let Φ and Λ_m be the set of all propositional variables and the set of all formulae. A logical consequence relation $\models$ is a binary relation on Λ_m.

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We now introduce a *sequent calculus* (L_{DmD}) as follows.

$$\frac{}{\phi \Rrightarrow \phi}\ \text{(Ax)} \qquad \frac{}{\neg\neg\phi \Rrightarrow \phi}\ \text{(DN-l)} \qquad \frac{}{\bot \Rrightarrow \phi}\ \text{(bot)}$$

$$\frac{}{\phi \Rrightarrow \top}\ \text{(top)} \qquad \frac{\phi \Rrightarrow \psi \quad \psi \Rrightarrow \chi}{\phi \Rrightarrow \chi}\ \text{(cut)} \qquad \frac{}{\Box\bot \Rrightarrow \bot}\ \text{(bot-}\Box\text{)}$$

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Definition

By a **partial semi-tense De Morgan algebra** is meant a triple $\mathbf{D} = (\mathbf{A}; G, F)$ such that $\mathbf{A} = (A; \leq, ', 0, 1)$ is a De Morgan poset with negation $'$ and G is a partial mapping of A into itself satisfying

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If $(\mathbf{A}_1; G_1, F_1)$ and $(\mathbf{A}_2; G_2, F_2)$ are partial semi-tense algebras, then a **morphism of partial semi-tense algebras** $f\colon (\mathbf{A}_1; G_1, F_1) \to (\mathbf{A}_2; G_2, F_2)$ is a morphism of De Morgan posets such that $f(G_1(a)) = G_2(f(a))$, for any $a \in A_1$ such $G_1(a)$ is defined and $f(F_1(b)) = F_2(f(b))$, for any $b \in A_1$ such $F_1(b)$ is defined, i.e., it simultaneously commutes with the respective semi-tense operators.

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A *frame* is a triple (S,T,R) where S,T are non-empty sets, $R \subseteq S \times T$. If $S = T$ we say that the pair (T,R) is a *time frame*. We say that

- (S,T,R) is *serial* if for all $s \in S$ there is $t \in T$ such that $s\,R\,t$.
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Theorem

Let $\mathbf{M}$ *be a complete De Morgan lattice,* (S,T,R) *be a frame,* $\widehat{G},\widehat{F}$ *be mappings from* M^T *into* M^S, *and* $\widehat{H},\widehat{P}$ *be mappings from* M^S *into* M^T *defined by*

$$\begin{aligned}
\widehat{G}(p)(s) &= \bigwedge\{p(t) \mid t \in T, s\,R\,t\},\\
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\end{aligned}$$

for all $p \in M^T$, $q \in M^S$, *and* $s \in S$, $t \in T$. *Then* $(\widehat{P},\widehat{G})$ *and* $(\widehat{F},\widehat{H})$ *are Galois connections such that,* $\widehat{F}(p) = \widehat{G}^{\partial}(p) = \widehat{G}(p')'$ *and* $\widehat{P}(q) = \widehat{H}^{\partial}(q) = \widehat{H}(q')'$. *If* (S,T,R) *is serial (coserial) then* $\widehat{G} \leq \widehat{F}$ $(\widehat{H} \leq \widehat{P})$. *Moreover, if* $S = T$ *and* (T,R) *is serial (coserial) then* $\widehat{G}$ *and* $\widehat{P}$ $(\widehat{H}$ *and* $\widehat{F})$ *are* semi-tense operators *in our sense, and* $(\mathbf{M}^T;\widehat{G},\widehat{F})$ $((\mathbf{M}^T;\widehat{H},\widehat{P}))$ *is a* complete semi-tense De Morgan algebra.

If $S = T$ we say that the **operators** $\widehat{G}$ **and** $\widehat{F} = \widehat{G}^{\partial}$ **on** M^T **are constructed by means of** (T,R). If (T,R) is serial and $\mathbf{M} = \mathbf{M}_2$ we say that $(\mathbf{M}_2^T;\widehat{G},\widehat{G}^{\partial})$ is a **complex semi-tense De Morgan algebra constructed by means of** (T,R).

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Theorem

Let $\mathbf{M}$ *be a complete De Morgan lattice,* (S,T,R) *be a frame,* $\widehat{G},\widehat{F}$ *be mappings from* M^T *into* M^S*, and* $\widehat{H},\widehat{P}$ *be mappings from* M^S *into* M^T *defined by*

$$\begin{array}{rcl}\widehat{G}(p)(s) & = & \bigwedge\{p(t) \mid t\in T, s\,R\,t\},\\ \widehat{F}(p)(s) & = & \bigvee\{p(t) \mid t\in T, s\,R\,t\},\\ \widehat{H}(q)(t) & = & \bigwedge\{q(s) \mid t\in T, s\,R\,t\},\\ \widehat{P}(q)(t) & = & \bigvee\{q(s) \mid t\in T, s\,R\,t\}\end{array}$$

for all $p\in M^T$, $q\in M^S$*, and* $s\in S$, $t\in T$. *Then* $(\widehat{P},\widehat{G})$ *and* $(\widehat{F},\widehat{H})$ *are Galois connections such that,* $\widehat{F}(p)=\widehat{G}^{\partial}(p)=\widehat{G}(p')'$ *and* $\widehat{P}(q)=\widehat{H}^{\partial}(q)=\widehat{H}(q')'$. *If* (S,T,R) *is serial (coserial) then* $\widehat{G}\leq\widehat{F}$ $(\widehat{H}\leq\widehat{P})$. *Moreover, if* $S=T$ *and* (T,R) *is serial (coserial) then* $\widehat{G}$ *and* $\widehat{P}$ $(\widehat{H}$ *and* $\widehat{F})$ *are* semi-tense operators *in our sense, and* $(\mathbf{M}^T;\widehat{G},\widehat{F})$ $((\mathbf{M}^T;\widehat{H},\widehat{P}))$ *is a* complete semi-tense De Morgan algebra.

If $S=T$ we say that the **operators** $\widehat{G}$ **and** $\widehat{F}=\widehat{G}^{\partial}$ **on** M^T **are constructed by means of** (T,R). If (T,R) is serial and $\mathbf{M}=\mathbf{M}_2$ we say that $(\mathbf{M}_2^T;\widehat{G},\widehat{G}^{\partial})$ is a **complex semi-tense De Morgan algebra constructed by means of** (T,R).

The construction of time frames and induced relations

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ be a De Morgan poset, $G\colon A \to A$ a semi-tense operator on $\mathbf{A}$. Let us put

$$R_G = \{(s,t) \in T_{\mathbf{A}}^{\mathsf{DMP}} \times T_{\mathbf{A}}^{\mathsf{DMP}} \mid (\forall x \in dom(G))(s(G(x)) \leq t(x))\}.$$

The construction of time frames and induced relations

Theorem

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ *be a De Morgan poset,* $G\colon A \to A$ *a semi-tense operator on* $\mathbf{A}$*. Then* $(T_{\mathbf{A}}^{DMP}, R_G)$ *is a time frame. Let* $\widehat{G}$ *be the operator constructed by means of the time frame* $(T_{\mathbf{A}}^{DMP}, R_G)$ *and let us put* $F = G^{\partial}$ *and* $\widehat{F} = \widehat{G}^{\partial}$*. Then the mapping* $i_{\mathbf{A}}$ *is an order-reflecting morphism of De Morgan posets into the complete De Morgan lattice* $\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}}$ *such that the following diagram commutes whenever the respective compositions are defined:*

$$\begin{array}{ccccc}
A & \xleftarrow{\;F\;} & A & \xrightarrow{\;G\;} & A \\
\downarrow{\scriptstyle i_{\mathbf{A}}} & & \downarrow{\scriptstyle i_{\mathbf{A}}} & & \downarrow{\scriptstyle i_{\mathbf{A}}} \\
\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}} & \xleftarrow[\widehat{F}]{} & \mathbf{M}_2^{T_{\mathbf{A}}^{DMP}} & \xrightarrow[\widehat{G}]{} & \mathbf{M}_2^{T_{\mathbf{A}}^{DMP}}
\end{array}$$

Theorem

(Soundness of $\mathbf{L}_{DmD}$). The normal D-modal poset-based logic for the De Morgan negation $\mathbf{L}_{DmD}$ is sound for the class of semi-tense De Morgan algebras. That is, for every sequent $\phi \mapsto \psi$ in $\mathbf{L}_{DmD}$, $s_\phi \leq t_\psi$ is valid on all semi-tense De Morgan algebras, where s_ϕ and t_ψ are the corresponding term functions for ϕ and ψ.

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(Completeness of $\mathbf{L}_{DmD}$). The normal D-modal poset-based logic for the De Morgan negation $\mathbf{L}_{DmD}$ is complete with respect to the class of semi-tense De Morgan algebras.

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Remark

Similarly as fo De Morgan posets one can easily check that the Lindenbaum-Tarski algebra $\mathbb{A}_{DmD}$ is a free semi-tense De Morgan algebra over the set of propositional variables, i.e., for any mapping e from the set Φ of propositional variables to a semi-tense De Morgan algebra $\mathbf{A}$ there is a uniquely determined morphism of semi-tense De Morgan algebras $\widetilde{e} : \mathbb{A}_{DmD} \to \mathbf{A}$ such that $e(p) = \widetilde{e}([p]_{\equiv})$ for all propositional variables p.

Theorem (Completeness Theorem for complex semi-tense De Morgan algebras)

An equation holds in every complex semi-tense De Morgan algebra if and only if it holds in every semi-tense De Morgan algebra.

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Outline

The so-called finite model property and the decidability are in logic considered as very appropriate properties which enable to use this logic in numerous applications both inside and outside the traditional logical reasoning. It can be of some interest that all the logics mentioned here have these very strong properties.

Theorem

The poset-based logic for the De Morgan negation $\mathbf{L}_{DMP}$ has the finite model property, i.e., any formula that fails in a model of $\mathbf{L}_{DMP}$ fails in a finite model.

Idea of proof: use $\mathbf{M}_2$.

Theorem

The poset-based logic for the De Morgan negation $\mathbf{L}_{DMP}$ is decidable, i.e., for any formula it is possible to decide, in a finite number of steps, whether it is a theorem or not.

Proof.

Let ϕ be a formula. We have an enumeration $\mathfrak{E}$ of all formal proofs in our axiomatic system. We can now proceed to decide whether ϕ is a theorem or not. If ϕ is not a theorem then it fails in $\mathbf{M}_2$ and this task can be accomplished in a finite number of steps. If ϕ is a theorem then there is a formal proof of ϕ. This formal proof must occur in our enumeration $\mathfrak{E}$. □

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Theorem

The normal D-modal poset-based logic for the De Morgan negation $\mathbf{L}_{DmD}$ has the finite model property.

Idea of proof: use so called filtration method.

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Note that one can extend the number of semi-tense operators finitely many times and our results remain valid.

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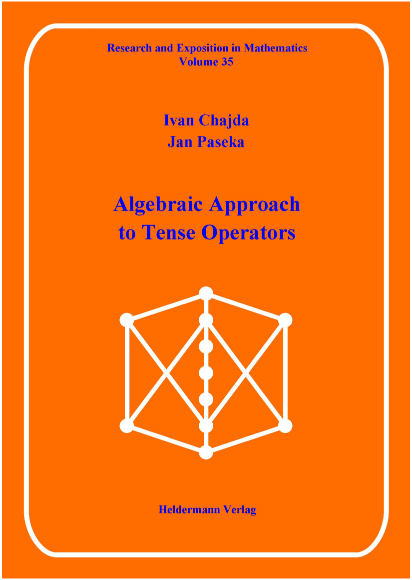

Thank you for your attention.