Distributivity of a Segmentation Lattice

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The problem

Terminological convention

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The problem is simple stated.

Problem

Characterize all distributive spaces.

The explanation

What does it mean?

- A *closures space with closed singletons* (*CS-space*) is a set (*the base set*) together with a system of subsets (*closed sets*) which
  1. is closed under all intersections
  2. contains all singletons
- A *segmentation* of a given CS-space is a partition of the base set into the closed sets.

Important facts

Canonical segmentation

Given a CS-space $(A, \mathcal{C})$, each closed set X induces a segmentation $\lambda(X)$ whose only possible nontrivial class is X. Clearly $\lambda\{x\} = \Delta$ for each $x \in A$.

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Proposition

The set of all segmentations $\mathcal{S}(\mathcal{C})$ of a given CS-space $(A, \mathcal{C})$ on a set A forms a complete lattice which is a sub-$\wedge$-semilattice of the lattice $\mathbf{Part}(A)$ *of all partitions on A.*

The motivation

Why do we ask that?

Inner structure on a CS-space

Given a CS-space $(A, \mathcal{C})$, a there is a canonical generalized ultrametric ω on A with values in $\mathcal{S}(\mathcal{C})$ compatible with $\mathcal{C}$ in such a way that closed sets coincide with closed balls w.r.t. ω

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Application of distributive lattice valued GUM

Given generalized ultrametric space A with values in a complete lattice L, then there is multiplicative structure on A. If L is **distributive** lattice, then there exists an adjoint operation of a scalar division which enables an algorithm for finding a segmentation based upon connection of initial pairs of elements.

Tree spaces

Example

Let (V, E) be a tree and let $\mathcal{C}$ be a set of all connected subsets. Then $(V, \mathcal{C})$ is a CS-space, which is, moreover, distributive.

A simple reason is that segmentations of the tree are in one-to-one correspondence with the subsets of E. Thus $\mathcal{S}(\mathcal{C}) \cong \mathcal{P}(E)$.

Centralized spaces

The essence of the above property is in the non-presence of cycles in the graph. The following weaker property is satisfied by all distributive spaces.

Definition

We say a CS-space $(A, \mathcal{C})$ is *centralized* if for each $x, y, z \in A$

$$\mathrm{cl}\{x, y\} \cap \mathrm{cl}\{x, y\} \cap \mathrm{cl}\{y, z\} \neq \emptyset.$$

An element in the above intersection is called *central* for the set $\{x, y, z\}$.

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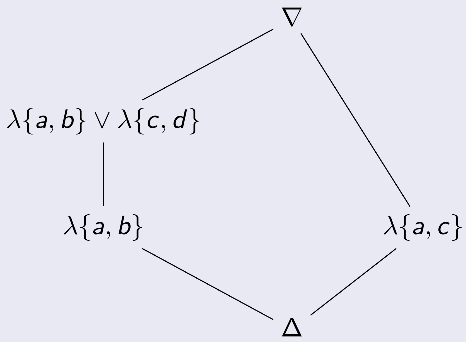
Centralized nondistributive space

Example

Let A be a square $\{a, b, c, d\}$ whose closed proper subsets are singletons and edges $\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}$. It is a CS-space and among any three elements there exists one between the other two. Thus the space is centralized but it is not distributive. The lattice $\mathcal{S}(\mathcal{C})$ contains the sublattice isomorphic to $\mathcal{N}_3$:



Segmentational cycles

Definition

- A (segmentational) *cycle* S in a CS-space $(A, \mathcal{C})$ is sequence of pairs $(x_i, \alpha_i)_{i=0}^{k}$ where $x_i \in A$ and $\alpha_i \in \mathcal{S}(\mathcal{C})$ for each $i \in \{1, \ldots, k\}$, $(x_0, \alpha_0) = (x_k, \alpha_k)$ and it satisfies $(x_i, x_{i+1}) \in \alpha_i$ for each $i \in \{1, \ldots, k\}$.

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- For $k = 3$ we have a *triangle*, for $k = 4$ *square*, etc.
- Two cycles $(x_i, \alpha_i)_{i=1}^k$, $(y_i, \beta_i)_{i=1}^l$ are *compatible*, if $k = l$ and for every $i \in \{1, \ldots, k\}$

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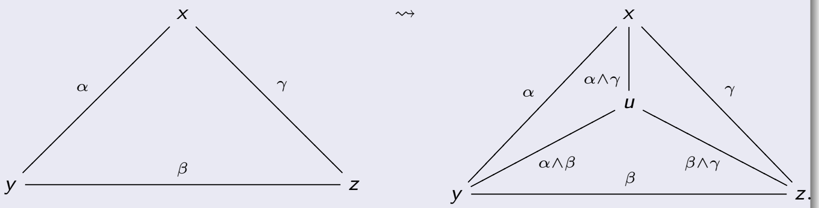
- A cycle is *linked* if $x_i \in \mathrm{cl}\{x_{i-1}, x_{i+1}\}$ for each $i \in \{1, \ldots, k\}$.

Central elements

Lemma

Let $u \in A$ be central for the elements of a triangle $(x, \gamma), (y, \beta), (z, \alpha)$. Then we have 3 more cycles:



Compatible linked cycles

Lemma

For each cycle in a finite centralized CS-space there exists a compatible linked cycle.

Sketch of the proof: Given a cycle $(x_i, \alpha_i)_{i=0}^{k}$ we extend the sequence for $i \in \mathbb{N}_0$ such that x_{i+k+1} be a central element for the triple $(x_{i+k}, x_{i+1}, x_{i+2})$ for each i. It will follow that $\mathrm{cl}\{x_i, x_{i+1}\} \supseteq \mathrm{cl}\{x_{i+k}, x_{i+k+1}\}$. Due to finiteness we end up with the same sets which enables to choose central elements such that $x_{i+k} = x_i$ for i larger than some bound. This will produce a cycle which is linked, due to the centrality of each of its elements.

Connectivity

A *connection* on a set is a system of subset (which are called *connected*) which satisfies that the system is closed under unions of systems with nonempty intersection.

Connective space

A CS-space is called *connective*, if the closed sets form a connection.

As a direct consequence of connectivity: for each three elements x, y, z

$$\mathrm{cl}\{x, z\} \subseteq \mathrm{cl}\{x, y\} \cup \mathrm{cl}\{y, z\}$$

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The notion of connection was introduced by J. Serra who observed that the space is connective iff the suprema in $\mathcal{S}(\mathcal{C})$ correspond to those in $\mathbf{Part}(A)$.

Consequences of connectivity

Given $\alpha, \beta \in \mathcal{S}(\mathcal{C})$, $(x, y) \in \alpha \vee \beta \Leftrightarrow y$ is an endpoint of a sequence of elements y_i such that, $y_0 = x$, $y = y_m$ and $(y_{2k}, y_{2k+1}) \in \alpha$ and $(y_{2k+1}, y_{2k+2}) \in \beta$, $k \in \{0, \ldots, m\}$. We may use a graph interpretation of the space with two kinds of edges (α, β), then $(x, y) \in \alpha \vee \beta$ iff there exists a path from x to y containing only two kinds of edges: α and β; such a path will be called (α, β)-path.

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Lemma

Let $(A, \mathcal{C})$ be an centralized connective CS-space. Then each element of a linked cycle belongs to the closure of a pair of two other neighboring elements in the cycle.

Distributivity

Recall a *relative complement* to an element b of a lattice L in an interval $[m, n] \subseteq L$ is an element $a \in [m, n]$ such that $a \wedge b = m$ and $a \vee b = n$. It is well known that the distributivity is equivalent to the uniqueness of the relative complements to every b in every interval.

Theorem

Every finite connective centralized CS-space is distributive.

Chain of implications

We have obtained:

finite connective centralized $\Rightarrow$ distributive $\Rightarrow$ centralized

with no arrow reversible (see bellow).

Sketch of the proof of the theorem

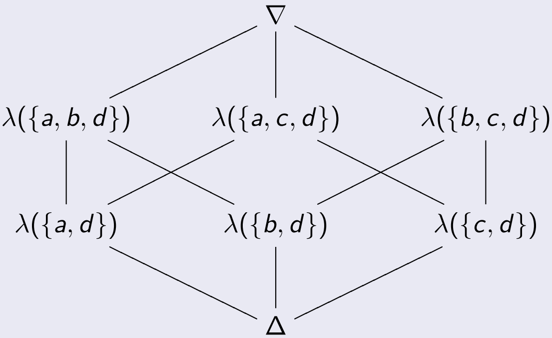
We assume an existence of two α, γ are two relative complements of β in the lattice of segmentations on a space $(A, \mathcal{C})$. We show that each pair $(x, y) \in \alpha$ belongs to γ, which then implies, by symmetry, $\alpha = \gamma$. Since $\alpha \leq \beta \vee \gamma$, due to the connectivity there exists a (β, γ)-path from x to y. Then one shows that

- If the path has the length 1, then immediately $(x, y) \in \gamma$.
- If the shortest path has the length $l > 1$, then it can be shortened using the previous lemma, thus we get a contradiction.

Reversed direction

Example

Let $A = \{a, b, c, d, e\}$ and $\mathcal{C}$ contains, beside A and the singletons, the sets $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. Then $(A, \mathcal{C})$ is not connective since $\{a, b, c, d\} = \{a, b, d\} \cup \{a, c, d\}$ is not closed but it is a union of two intersecting sets. However, the lattice $\mathcal{S}(\mathcal{C})$ is distributive since the Hasse diagram is:



Thank you for your attention