Congruence FD-maximal algebras

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Congruence lattices

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

In this lecture we concentrate on

Problem. Let $\mathcal{K}$ be a finitely generated CD variety. Describe finite members of Con $\mathcal{K}$.

Necessary condition

In the sequel: $\mathcal{V}\dots$ a finitely generated CD variety;
SI$(\mathcal{V})\dots$ the family of subdirectly irreducible members;
$\mathrm{M}(L)\dots$ completely $\wedge$-irreducible elements of a lattice L.

Lemma

Let $L \in \mathrm{Con}\,\mathcal{V}$. Then for every $x \in \mathrm{M}(L)$, the lattice $\uparrow x$ is isomorphic to $\mathrm{Con}\,T$ for some $T \in \mathrm{SI}(\mathcal{V})$.

Congruence FD-maximal varieties

On the finite level (for finite L), the necessary condition is sometimes also sufficient. In such a case we say that $\mathcal{V}$ is *congruence FD-maximal*. Formally, $\mathcal{V}$ is congruence FD-maximal, if for every *finite distributive* lattice L the following two conditions are equivalent:

(i) $L \in \operatorname{Con} \mathcal{V}$;

(ii) for every $x \in \mathrm{M}(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in \mathrm{SI}(\mathcal{V})$.

In other words, $\mathcal{V}$ is congruence FD-maximal iff the class of all finite members of $\operatorname{Con} \mathcal{V}$ is as large as possible by the necessary condition.

Congruence FD-maximal algebras

Let A be a finite subdirectly irreducible algebra generating a CD variety. We say that A is *congruence FD-maximal*, if for every finite distributive lattice L the following two conditions are equivalent:

(i) $L \in \operatorname{Con} \mathrm{P}_s \mathrm{H}(A)$;

(ii) for every $x \in \mathrm{M}(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in \mathrm{H}(A)$.

In other words, A is congruence FD-maximal iff the class of all finite members of $\operatorname{Con} \mathrm{P}_s \mathrm{H}(A)$ is as large as possible by the necessary condition.

Connection

Theorem

If $\mathcal{V}$ is congruence FD-maximal, then there is a family $\mathcal{M} \subseteq \mathrm{SI}(\mathcal{V})$ such that

(i) *every $B \in \mathcal{M}$ is congruence FD-maximal;*

(ii) *for every $A \in \mathrm{SI}(\mathcal{V})$ there is $B \in \mathcal{M}$ with $\mathrm{Con}\, A \cong \mathrm{Con}\, B$;*

(iii) *if $A, B \in \mathcal{M}$, $\alpha \in \mathrm{Con}\, A$, $\beta \in \mathrm{Con}\, B$ with $\uparrow\alpha \cong \uparrow\beta$, then $A/\alpha \cong B/\beta$.*

Proof: Ramsey type argument

Conjecture: The converse holds

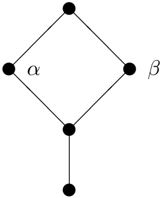
Easy case

Theorem

Every finite algebra generating a CD variety, whose congruence lattice is a chain, is congruence FD-maximal.

The simplest of the difficult cases

Let A be a finite algebra generating CD variety such that $\operatorname{Con} A$ is



Compatible families

Let E be a subset of $B \times B$ for some set B. Let X be a set and let $\mathcal{F}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is *E-compatible* if $\{(f(x), g(x)) \mid x \in X\} = E$ or $\{(g(x), f(x)) \mid x \in X\} = E$ for every $f, g \in \mathcal{F}$, $f \neq g$.

Lemma

Suppose that $E \subseteq B \times B$ contains a pair (a, b) with $a \neq b$. Then the following condition are equivalent.

(i) *There exist arbitrarily large finite E-compatible sets of functions.*

(ii) *For every $(a, b) \in E$ there are $x, y, z \in B$ such that* $(x, x), (y, y), (z, z), (x, y),$ $(x, z), (y, z), (x, a), (x, b), (a, y), (y, b), (a, z), (b, z) \in E.$

Characterization theorem

Let A be as above.

Theorem

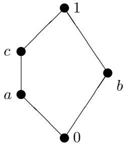
A is congruence FD-maximal iff

(i) *A/α and A/β are isomorphic to the same algebra B;*

(ii) *there are homomorphisms $h_0, h_1 : A \to B$ with $\mathrm{Ker}(h_0) = \alpha$, $\mathrm{Ker}(h_1) = \beta$ such that the relation $E = \{(h_0(x), h_1(x)) \mid x \in A\} \subseteq B \times B$ admits arbitrarily large E-compatible sets of functions.*

Positive example

For $A = N_5$ we have $B = \{0, 1\}$, $E = \{(0,0),(0,1),(1,0),(1,1)\}$ so almost every family of functions is E-compatible and A is congruence FD-maximal.



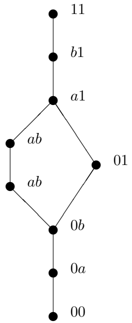
N_5

Negative example 1

The lattice N_5 with the distinguished element (nullary operation) b is not congruence FD-maximal, because the quotients N_5/α and N_5/β are not isomorphic.

Negative example2

Consider the following lattice A with two additional unary operations.



$f(00) = 00, f(0a) = 0b, f(0b) = f(01) = 01$

$f(ab) = f(a1) = b1, f(b1) = f(11) = 11$

$g(11) = 11, g(b1) = a1, g(a1) = g(01) = 01$

$g(ab) = g(0b) = 0a, g(0a) = g(00) = 00$

Negative example 2

We have $B = \{0, 1, a, b\}$,
$E = \{(0, 0), (0, a), (0, b), (a, b), (0, 1), (a, 1), (b, 1), (1, 1)\}$ (the labels on the elements of A), and the pair (a, b) violates the condition. Thus, A is not congruence FD-maximal.

Generalization

Let $\mathrm{Con}\, A$ be the n-dimensional cube with a new zero added. Let $\alpha_1, \ldots, \alpha_n$ be the coatoms of $\mathrm{Con}\, A$

The concept of a compatible family has to be generalized.

Let E be a subset of B^n for some set B. For a permutation π on $\{1, \ldots, n\}$ denote

$$E^\pi = \{(a_{\pi(1)}, \ldots, a_{\pi(n)}) \mid (a_1, \ldots, a_n) \in E\}.$$

Let X be a set and let $\mathcal{F}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is *E-compatible* if for every mutually distinct $f_1, \ldots, f_n \in \mathcal{F}$ there exists π such that $\{(f_1(x), \ldots, f_n(x)) \mid x \in X\} = E^\pi$.

Compatible families generalized

Lemma

Suppose that $E \subseteq B^n$ contains a non-diagonal n-tuple. Then the following condition are equivalent.

(i) *There exist arbitrarily large finite E-compatible sets of functions.*

(ii) *There exist a permutation π such that for every $(a_2, a_4, \dots, a_{2n}) \in E^\pi$ there are $a_1, a_3, \dots, a_{2n+1} \in B$ such that $(a_{i_1}, \dots, a_{i_n}) \in E^\pi$ whenever $i_1 \leq \dots \leq i_n$ and every every even k appears at most once among $i_1, \dots, i_n$.*

Characterization theorem

Let A be as above.

Theorem

A is congruence FD-maximal iff

(i) *all quotients A/α_i are isomorphic to the same algebra B;*

(ii) *there are homomorphisms $h_i : A \to B$ with $\mathrm{Ker}(h_i) = \alpha_i$, $(i = 1, \dots, n)$ such that the relation $E = \{(h_1(x), \dots, h_n(x)) \mid x \in A\} \subseteq B^n$ admits arbitrarily large E-compatible sets of functions.*

General case

Again, quotients with isomorphic congruence lattices must themselves be isomorphic. Actually, we can prove more.

Theorem

Let A be congruence FD-maximal, $\alpha, \beta \in \operatorname{Con} A$ with $\uparrow\alpha \cong \uparrow\beta$. Then there is an isomorphism $\varphi : \uparrow\alpha \to \uparrow\beta$ and isomorphisms $f_\gamma : A/\gamma \to A/\varphi(\gamma)$ for all $\gamma \in \uparrow\alpha$ that commute with the natural projections, that is

$$\begin{array}{ccc} A/\gamma & \xrightarrow{f_\gamma} & A/\varphi(\gamma) \\ \pi_{\gamma\delta}\downarrow & & \downarrow\pi_{\varphi(\gamma)\varphi(\delta)} \\ A/\delta & \xrightarrow{f_\delta} & A/\varphi(\delta) \end{array}$$

whenever $\gamma \leq \delta$.

What remains to do?

Problem. Characterize heterogeneous relations that admit arbitrarily large systems of compatible functions.