Going Up and Lying Over in Congruence–modular Algebras

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Introduction

Study of Properties **Going Up** and **Lying Over**:

- originated in ring theory;
- studied for field extensions;
- studied in class field theory;
- more recently, studied in MV–algebras and abelian lattice–ordered groups;
- here we initiate the study of these properties in the general setting of **congruence–modular algebras**.

These are algebras of many–valued logics

Definition

(Commutative) residuated lattice: algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$, where:

- $(A, \vee, \wedge, 0, 1)$: bounded lattice (with partial order $\leq$)
- $(A, \odot, 1)$: commutative monoid
- $\rightarrow$: binary operation on A which satisfies the *law of residuation:* for all $a, b, c \in A$, $a \leq b \rightarrow c$ iff $a \odot b \leq c$

- for instance, any finite distributive lattice L can be organized as a residuated lattice, with $\odot = \wedge$ and $x \rightarrow y = \max\{u \in L \mid u \wedge x \leq y\}$ for all $x, y \in L$
- and any Boolean algebra can be organized as a residuated lattice, with $\odot = \wedge$ and $\rightarrow$ equal to the Boolean implication

- MV–algebras form a subclass of the class of residuated lattices; moreover, so do BL–algebras and MTL–algebras
- the category of MV–algebras is equivalent to that of abelian lattice–orderred groups with strong unit
- residuated lattices form an equational class; so do MV–algebras, and all the algebras we are referring to in this presentation (with the obvious exception of non–distributive lattices)

Some notations

- I: a non–empty set
- $(M_i)_{i \in I}$, $(N_i)_{i \in I}$: families of sets
- for all $i \in I$, $f_i : M_i \to N_i$
- $\prod_{i \in I} f_i : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$, for all $(a_i)_{i \in I} \in \prod_{i \in I} M_i$, $(\prod_{i \in I} f_i)((a_i)_{i \in I}) = (f_i(a_i))_{i \in I}$

- M and N: sets
- $f : M \to N$ (so $f^2 = f \times f : M^2 \to N^2$)
- $U \subseteq M$, $V \subseteq N$
- $f(U) = \{f(a) \mid a \in U\}$, $f^{-1}(V) = \{a \in M \mid f(a) \in V\}$
- $X \subseteq M^2$, $Y \subseteq N^2$
- $f(X) = f^2(X) = \{(f(a), f(b)) \mid (a, b) \in X\}$,
  $f^*(Y) = (f^2)^{-1}(Y) = \{(a, b) \in M^2 \mid (f(a), f(b)) \in Y\}$

- for any $n \in \mathbb{N}^*$, $\mathcal{L}_n$ = the n–element chain
- $\mathcal{D}$ = the diamond
- $\mathcal{P}$ = the pentagon

Algebras; lattice of congruences; generated congruences

- any algebra shall be designated by its support set
- any algebra shall be considerred non–empty
- *trivial algebra*: one–element algebra
- *non–trivial algebra*: algebra with at least two distinct elements

- A : an algebra

- $(\mathrm{Con}(A), \vee, \cap, \Delta_A, \nabla_A)$: the bounded lattice of congruences of A: a complete lattice
- A : *congruence–modular* iff the lattice $\mathrm{Con}(A)$ is modular
- A : *congruence–distributive* iff the lattice $\mathrm{Con}(A)$ is distributive

- for all $\theta \in \mathrm{Con}(A)$, $[\theta) = \{\alpha \in \mathrm{Con}(A) \mid \theta \subseteq \alpha\}$

- for all $X \subseteq A^2$: $Cg_A(X) =$ the congruence of A generated by X
- for all $a, b \in A$: $Cg_A(a, b) = Cg_A(\{(a, b)\})$: the principal congruence of A generated by (a, b)

Morphisms and congruences in classes of algebras

- $\mathcal{C}$: a class of algebras of the same type

- A and B: algebras in $\mathcal{C}$
- $f : A \to B$: a morphism in $\mathcal{C}$
- for all $\alpha \in \mathrm{Con}(A)$, $f(\alpha) \in \mathrm{Con}(f(A))$
- for all $\beta \in \mathrm{Con}(B)$, $f^*(\beta) \in \mathrm{Con}(A)$
- the *kernel* of f: $\mathrm{Ker}(f) = f^*(\Delta_B) \in \mathrm{Con}(A)$

- $\mathcal{C}$: *congruence–modular* iff each algebra in $\mathcal{C}$ is congruence–modular; example: the class of commutative unitary rings
- $\mathcal{C}$: *congruence–distributive* iff each algebra in $\mathcal{C}$ is congruence–distributive; examples: the classes of lattices, MV–algebras, residuated lattices
- $\mathcal{C}$: *semi–degenerate* iff no non–trivial algebra in $\mathcal{C}$ has trivial subalgebras, or, equivalently, iff, for all algebras A in $\mathcal{C}$, $\nabla_A = A^2$ is a finitely generated congruence; examples: the classes of unitary rings, bounded lattices, MV–algebras, residuated lattices

The commutator, and prime congruences

- $\mathcal{C}$: a congruence–modular equational class of algebras of the same type
- hence in $\mathcal{C}$ there exists the *commutator*: for every algebra A in $\mathcal{C}$, $[\cdot,\cdot]_A : \mathrm{Con}(A) \times \mathrm{Con}(A) \to \mathrm{Con}(A)$, such that, for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha, \beta]_A$ is the least of the congruences $\mu \in \mathrm{Con}(A)$ with these properties:
  1. $\mu \subseteq \alpha \cap \beta$
  2. for any algebra B from $\mathcal{C}$ and any surjective morphism in $\mathcal{C}$ $f : A \to B$, $\mu \vee \mathrm{Ker}(f) = f^*([f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]_B)$
- we recall that the commutator is unique (for any congruence–modular equational class, in each of its members)
- if $\mathcal{C}$ is congruence–distributive, then, in every member A of $\mathcal{C}$, the commutator, $[\cdot,\cdot]_A$, equals the intersection of congruences
- A: an algebra in $\mathcal{C}$
- $\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\}$
- ϕ is called a *prime congruence* iff, for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha, \beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$
- $\mathrm{Spec}(A)$ = the set of the prime congruences of A

Definitions

From now on:

- $\mathcal{C}$: a semi–degenerate congruence–modular equational class of algebras of the same type
- A and B: algebras in $\mathcal{C}$
- $f : A \to B$: a morphism in $\mathcal{C}$

Definition

- We call f an *admissible morphism* iff, for all $\psi \in \mathrm{Spec}(B)$, we have $f^*(\psi) \in \mathrm{Spec}(A)$ (that is $f^*(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$).

Now assume that f is admissible.

- We say that f fulfills the *Going Up* property (abbreviated *GU*) iff, for any $\phi, \psi \in \mathrm{Spec}(A)$ and any $\phi_1 \in \mathrm{Spec}(B)$ such that $\phi \subseteq \psi$ and $f^*(\phi_1) = \phi$, there exists a $\psi_1 \in \mathrm{Spec}(B)$ such that $\phi_1 \subseteq \psi_1$ and $f^*(\psi_1) = \psi$. Formally:
$$f \text{ fulfills } GU \text{ iff, for all } \phi_1 \in \mathrm{Spec}(B),$$
$$[f^*(\phi_1)) \cap \mathrm{Spec}(A) \subseteq f^*([\phi_1) \cap \mathrm{Spec}(B)).$$
- We say that f fulfills the *Lying Over* property (abbreviated *LO*) iff, for any $\phi \in \mathrm{Spec}(A)$ such that $\mathrm{Ker}(f) \subseteq \phi$, there exists a $\phi_1 \in \mathrm{Spec}(B)$ such that $f^*(\phi_1) = \phi$. Formally:
$$f \text{ fulfills } LO \text{ iff } [\mathrm{Ker}(f)) \cap \mathrm{Spec}(A) \subseteq f^*(\mathrm{Spec}(B)).$$

Admissible morphisms, Going Up and Lying Over

Proposition (GU implies LO)

If f is admissible and fulfills GU, then f fulfills LO.

Proposition

- Any surjective morphism is admissible, but the converse is not true.
- Not all morphisms are admissible.
- Not all admissible morphisms fulfill GU or LO.
- In the classes of Boolean algebras and residuated lattices, all morphisms are admissible and fulfill GU (thus also LO).

We shall provide examples for the negative statements above in the congruence–distributive (semi–degenerate) class of bounded lattices. Concerning the second and third statement, note, however, that many bounded lattice morphisms are admissible and fulfill GU (thus also LO):

- if L and M are bounded lattices such that M can be obtained through finite direct products and/or finite ordinal sums from finite distributive lattices and/or bounded chains and/or Boolean algebras, then any bounded lattice morphism $h : L \to M$ is admissible and fulfills GU (thus also LO).

An admissible morphism which is not surjective

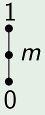
Example

$\mathrm{Spec}(\mathcal{L}_2) = \{\Delta_{\mathcal{L}_2}\}$.
$\mathrm{Spec}(\mathcal{L}_3) = \{\phi, \psi\}$, where $\mathcal{L}_3/\phi = \{\{0, m\}, \{1\}\}$ and $\mathcal{L}_3/\psi = \{\{0\}, \{m, 1\}\}$.
Let i be the canonical embedding of $\mathcal{L}_2$ into $\mathcal{L}_3$:

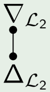
$\mathcal{L}_2$

$\xrightarrow{\ i\ }$

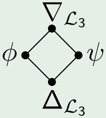
u	0	1
$i(u)$	0	1



$\mathcal{L}_3$



$\mathrm{Con}(\mathcal{L}_2)$



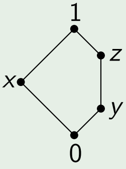
$\mathrm{Con}(\mathcal{L}_3)$

Clearly, i is a bounded lattice morphism which is not surjective.
$i^*(\phi) = i^*(\psi) = \Delta_{\mathcal{L}_2} \in \mathrm{Spec}(\mathcal{L}_2)$, therefore i is admissible.
Note, also, that i fulfills GU (thus also LO).

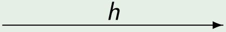
A morphism which is not admissible

Example

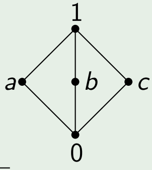
$\mathrm{Spec}(\mathcal{D}) = \{\Delta_{\mathcal{D}}\}$.
With $\mathcal{P}/\alpha = \{\{0, y, z\}, \{x, 1\}\}$, $\mathcal{P}/\beta = \{\{0, x\}, \{y, z, 1\}\}$ and $\mathcal{P}/\gamma = \{\{0\}, \{x\}, \{y, z\}, \{1\}\}$, we have $\mathrm{Spec}(\mathcal{P}) = \{\Delta_P, \alpha, \beta\}$. $\gamma \notin \mathrm{Spec}(\mathcal{P})$, because $\gamma = \alpha \cap \beta = [\alpha, \beta]_{\mathcal{P}} \supseteq [\alpha, \beta]_{\mathcal{P}}$, but $\alpha \nsubseteq \gamma$ and $\beta \nsubseteq \gamma$.



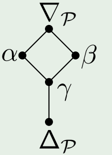
$\mathcal{P}$



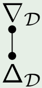
u	0	x	y	z	1
$h(u)$	0	a	b	b	1



$\mathcal{D}$



$\mathrm{Con}(\mathcal{P})$



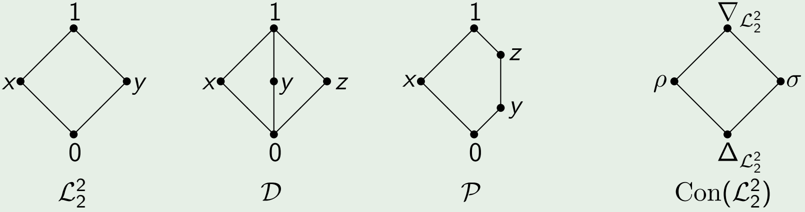
$\mathrm{Con}(\mathcal{D})$

Clearly, h is a bounded lattice morphism.
$h^*(\Delta_{\mathcal{D}}) = \gamma$, therefore h is not admissible.
Note, however, that all admissible morphisms from $\mathcal{P}$ to $\mathcal{D}$ fulfill GU (thus also LO).

Canonical embeddings which are not admissible

Example

Let $\mathcal{L}_2^2$, $\mathcal{D}$ and $\mathcal{P}$ have the elements denoted as in the following Hasse diagrams, i be the canonical embedding of $\mathcal{L}_2^2$ into $\mathcal{D}$ and j be the canonical embedding of $\mathcal{L}_2^2$ into $\mathcal{P}$:

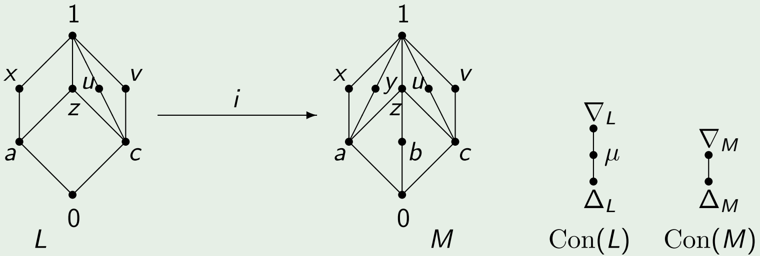


$\mathrm{Con}(\mathcal{L}_2^2) = \{\Delta_{\mathcal{L}_2^2}, \rho, \sigma, \nabla_{\mathcal{L}_2^2}\}$, where $\mathcal{L}_2^2/\rho = \{\{0, x\}, \{y, 1\}\}$ and $\mathcal{L}_2^2/\sigma = \{\{0, y\}, \{x, 1\}\}$, so $\mathrm{Spec}(\mathcal{L}_2^2) = \{\rho, \sigma\}$.
$\Delta_{\mathcal{D}} \in \mathrm{Spec}(\mathcal{D})$ and $i^*(\Delta_{\mathcal{D}}) = \Delta_{\mathcal{L}_2^2} \notin \mathrm{Spec}(\mathcal{L}_2^2)$, thus i is not admissible.
$\Delta_{\mathcal{P}} \in \mathrm{Spec}(\mathcal{P})$ and $j^*(\Delta_{\mathcal{P}}) = \Delta_{\mathcal{L}_2^2} \notin \mathrm{Spec}(\mathcal{L}_2^2)$, thus j is not admissible.

An admissible morphism without LO (thus without GU)

Example

L is a bounded sublattice of M. Let $i : L \to M$ be the canonical embedding.
$\mathrm{Spec}(M) = \{\Delta_M\}$ and $\mathrm{Spec}(L) = \{\Delta_L, \mu\}$, where
$L/\mu = \{\{0, a, x\}, \{c, z, u, v, 1\}\}$.



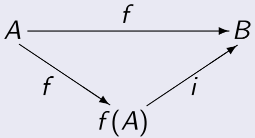
$i^*(\Delta_M) = \Delta_L$, thus i is admissible.
$\mathrm{Ker}(i) = i^*(\Delta_M) = \Delta_L \subseteq \mu \in \mathrm{Spec}(L)$ and there exists no $\phi \in \mathrm{Spec}(M) = \{\Delta_M\}$ such that $i^*(\phi) = \mu$, therefore i does not fulfill GU and it does not fulfill LO.

Embeddings are sufficient

From now on, whenever we state about a morphism that it fulfills GU or LO, we assume that it is admissible.

Proposition

Let $i : f(A) \to B$ be the canonical embedding.



Then the following hold:

- f is admissible iff i is admissible;
- f fulfills GU iff i fulfills GU;
- f fulfills LO iff i fulfills LO.

Preservation by composition and finite direct products

Proposition

Let C be an algebra in $\mathcal{C}$ and $g : B \to C$ be a morphism in $\mathcal{C}$. Then:

- if f and g are admissible, then $g \circ f$ is admissible;
- if f and g fulfill GU, then $g \circ f$ fulfills GU;
- if f and g fulfill LO, then $g \circ f$ fulfills LO.

Proposition

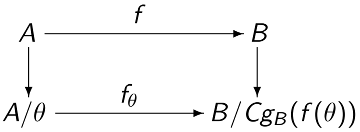
Assume that $\mathcal{C}$ has no skew congruences (so that, for any $n \in \mathbb{N}^*$ and any members $A_1, \ldots, A_n$ of $\mathcal{C}$, $\mathrm{Con}(\prod_{i=1}^{n} A_i) = \{\prod_{i=1}^{n} \theta_i \mid (\forall\, i \in \overline{1,n})\,(\theta_i \in \mathrm{Con}(A_i))\}$).

Let $n \in \mathbb{N}^*$ and, for all $i \in \overline{1,n}$, A_i, B_i be members of $\mathcal{C}$ and $f_i : A_i \to B_i$ be a morphism in $\mathcal{C}$. Let $f = \prod_{i=1}^{n} f_i : \prod_{i=1}^{n} A_i \to \prod_{i=1}^{n} B_i$. Then:

- f is admissible iff $f_1, \ldots, f_n$ are admissible;
- f fulfills GU iff $f_1, \ldots, f_n$ fulfill GU;
- f fulfills LO iff $f_1, \ldots, f_n$ fulfill LO.

Preservation by quotients

For all $\theta \in \mathrm{Con}(A)$, let $f_\theta : A/\theta \to B/Cg_B(f(\theta))$, defined by:
$f_\theta(a/\theta) = f(a)/Cg_B(f(\theta))$ for all $a \in A$.
Then each f_θ is a well–defined morphism in $\mathcal{C}$ and the following diagram is commutative, where the vertical arrows represent the canonical surjections:



Proposition

- f is admissible iff, for all $\theta \in \mathrm{Con}(A)$, f_θ is admissible;
- f fulfills GU iff, for all $\theta \in \mathrm{Con}(A)$, f_θ fulfills GU;
- f fulfills LO iff, for all $\theta \in \mathrm{Con}(A)$, f_θ fulfills LO.

Topological characterization for GU

If we denote, for all $\theta \in \mathrm{Con}(A)$, by $D_A(\theta) = \{\phi \in \mathrm{Spec}(A) \mid \theta \nsubseteq \phi\}$, then $\{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\}$ is a topology on $\mathrm{Spec}(A)$, called the *Stone topology*.

Proposition

If f is admissible, then the following are equivalent:

- f fulfills GU;
- the restriction $f^* \mid_{\mathrm{Spec}(B)}: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map with respect to the Stone topologies.

Classes of algebras with GU

Proposition

If $\mathcal{C}$ has *equationally definable principal congruences*, then every admissible morphism in $\mathcal{C}$ fulfills GU (thus also LO).

- The class of residuated lattices has equationally definable principal congruences. However, for this particular class, we have provided a direct proof for the fact that each of its morphisms is admissible and fulfills GU (thus also LO).
- See more examples of equational classes with equationally definable principal congruences in the work done by Blok, Köhler and Pigozzi.

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THANK YOU FOR YOUR ATTENTION!