Integral representation of states on quantum structures

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Introduction

Effect algebras (EAs) $(E; \oplus, 0, 1)$ (D.J. Foulis and M.K. Bennett, 1994) — abstract version of the system $\mathcal{E}(H)$ of all self-adjoint operators between zero and identity on a Hilbert space H – algebraic basis for (possibly) unsharp (fuzzy) quantum-mechanical measurements.

• Special subclasses: the class of orthoalgebras, orthomodular posets and orthomodular lattices, MV-algebras.

• Examples: (G, G^+) abelian po-group.
$\Gamma(G, a) := \{g \in G : 0 \leq g \leq a\}$, $a \in G^+$ is *interval effect algebra*.

• $\mathcal{E}(H)$, MV-algebras, effect algebras with RDP are interval effect algebras (*Mundici 1986*, *Ravindran 1996*).

LS-theorem, MV

Loomis-Sikorski theorem for σ MV-algebras:
$\forall\, \sigma$ MV-algebra M, $\exists$ tribe $\mathcal{T}$ of fuzzy sets on $\Omega \neq \emptyset$ and surjective σ MV-algebra morphism $h : \mathcal{T} \to M$ (*Mundici 1999, Dvurečenskij 2000*).

(1) $\forall f \in \mathcal{T}$, $f : \Omega \to [0,1] \subseteq \mathbb{R}$ is Borel measurable w.r. $\mathcal{B}_0(\mathcal{T}) := \{A \subseteq \Omega : \chi_A \in \mathcal{T}\}$

(2) $\forall$ σ-additive state s on M,

$$s(a) = \int_\Omega f(\omega) ds \circ h(\omega), a \in M, h(f) = a.$$

(*Butnariu,Klement 1993*).

LS-theorem, EA with RDP

L-S theorem for monotone σ-complete EAs with RDP: $\forall E\ \exists$ effect tribe $\mathcal{T}_e$ of functions $f : \Omega \to [0,1]$ and a surjective effect algebra σ-morphism $h : \mathcal{T}_e \to E$. (*Buhagiar, Chetcuti, Dvurečenskij 2006*).

• If (1) holds for $\mathcal{T}_e$, then also (2) holds for E. (*Dvurečenskij 2011*)

We show that if (1) holds, then $\mathcal{T}_e$ is a tribe and E represented by $\mathcal{T}_e$ is a σ MV-algebra.

Other consequence of (1) is a kind of *spectral theorem* for σ MV-algebras and *smearing* theorems for states and observables.

effect algebras

An *e*ffect algebra (EA) is a partial algebra $(E; \oplus, 0, 1)$ where E is a nonempty set, $\oplus$ is a partial binary operation and $0, 1$ are constants, such that

(E1) $a \oplus b = b \oplus a$ (commutativity);

(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

(E3) $\forall a \in E$ there is a unique $a' \in E$ with $a \oplus a' = 1$;

(E4) if $a \oplus 1$ is defined, then $a = 0$.

a and b are *orthogonal* ($a \perp b$) iff $a \oplus b$ is defined.
$a \leq b$ iff $\exists c \in E$: $a \oplus c = b$; then $c =: b \ominus a$,
$0 \leq a \leq 1$.

• E is *monotone σ-complete* iff every ascending sequence of elements in E has a supremum in E.

MV-(effect) algebras

E has *Riesz decomposition property* (RDP) iff whenever $a \leq b \oplus c$, there are $b_1 \leq b, c_1 \leq c$ with $a = b_1 \oplus c_1$.

• Every effect algebra E with RDP is a unit interval $G[0, u]$ in an abelian interpolation po-group (G, u) with strong unit u, where the partial operation $\oplus$ is defined be the restriction of the group operation $+$ to the unit interval (*Ravindran 1996*).

A lattice ordered effect algebra with RDP is an *MV-effect algebra*.

• MV-effect algebras are equivalent with MV-algebras.

• Every MV-algebra is a unit interval in an ℓ-group with a strong unit ((*Mundici 1986*).

effect tribe

An *effect tribe* is a system $\mathcal{T}_e \subseteq [0,1]^X$, $X \neq \emptyset$ such that:

(i) $1 \in \mathcal{T}_e$,

(ii) $1 - f \in \mathcal{T}_e$ whenever $f \in \mathcal{T}_e$,

(iii) $f + g \in \mathcal{T}_e$ whenever $f, g \in \mathcal{T}_e$ and $f \leq 1 - g$,

(iv) $(f_n)_n \subseteq \mathcal{T}_e$ and $f_n \nearrow f$ pointwise $\implies f \in \mathcal{T}_e$.

$\mathcal{T}_e$ is a monotone σ-complete EA with RDP.
Replacing (iii) and (iv) by

(iii') if $(f_n)_n$ is a sequence from $\mathcal{T}$, then $\min\{\sum_{i=1}^{\infty} f_n, 1\} \in \mathcal{T}$

we obtain a *tribe*. A tribe is a σ MV-algebra with the binary operation $f \boxplus g = \min\{f + g, 1\}$.

states, central elements

A *state* on an EA E is a mapping $s : E \to [0,1]$ such that (i) $s(1) = 1$; (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists.
A state s is σ-*additive* iff for every ascending sequence $a_n \nearrow a$ we have $s(a_n) \to s(a)$.
• $z \in E$ is *central* iff every $a \in E$ uniquely decomposes into $a = a_1 \oplus a_2$ with $a_1 \le z$, $a_2 \le z'$.
Central elements form a Boolean subalgebra of E.
If E has RDP, then central elements coincide with sharp elements of E ($a \in E$ is *sharp* iff $a \wedge a' = 0$).
• If $\mathcal{T}_e \subseteq [0,1]^X$ is an effect tribe, then sharp (central) elements are characteristic functions in $\mathcal{T}_e$, and $\mathcal{B}_0(\mathcal{T}_e) := \{A \subseteq X : \chi_A \in \mathcal{T}_e\}$ is a σ-algebra of sets.

integral representation

• Let $\mathcal{T}_e$ be an effect tribe such that $\forall f \in \mathcal{T}_e$, f is Borel measurable w.r. $\mathcal{B}_0(\mathcal{T}_e)$, then for every σ-additive state s on $\mathcal{T}_e$,

$$s(f) = \int_X f(x)\mu_s(dx),\ \ \mu_s(A) = s(\chi_A), A \in \mathcal{B}_0(\mathcal{T}).$$

• If E is monotone σ-complete EA with RDP with LS-representation $h : \mathcal{T}_e \to E$ such that $\mathcal{T}_e$ has the property that every $f \in \mathcal{T}_e$ is $\mathcal{B}_0(\mathcal{T}_e)$ measurable, then for every σ-additive state s on E,

$$s(a) = \int_X f(x)\mu_s(h(dx)),\ \ f \in \mathcal{T}_e, h(f) = a.$$

(*Dvurečenskij 2011*).

general comparability

- Effect tribe $\mathcal{T}_e$ with the property that $\forall f \in \mathcal{T}_e$, f is $\mathcal{B}_0(\mathcal{T}_e)$-measurable, has GC, hence is a lattice (*SP, Vinceková 2015*).

E satisfies *general comparability* property (GC), iff $\forall x, y \in E\ \exists e \in E$, e central: $x \wedge e \leq y \wedge e$, $x \wedge e' \geq y \wedge e'$ (*Goodearl 1986*).

- If E has GC, then E is a lattice.

($x, y \in E$, $x \wedge y = x \wedge e \oplus y \wedge e'$).

- $f, g \in \mathcal{T}_e$, $A := \{x \in X : f(x) \leq g(x)\}$, then $\chi_A \in \mathcal{T}_e$ is central. Then $f \wedge \chi_A \leq g \wedge \chi_A$, $f \wedge \chi_{A^c} \geq g \wedge \chi_{A^c}$.
- Effect tribe which is a lattice, is a tribe.

theorem

- Let E be a monotone σ-complete EA with RDP and let $(X, \mathcal{T}_e, h)$ be a LS-representation of E with property (1), i.e. such that every $f \in \mathcal{T}_e$ is $\mathcal{B}_0(\mathcal{T}_e)$-measurable. Then E is a lattice, thus a σ MV-algebra (*SP, Vinceková 2015*).

spectral theorem

M - σ MV-algebra,
$(X, \mathcal{T}, h)$ its LS-representation.

- To every $a \in M$, there is a sharp observable

$$\Lambda_a : \mathcal{B}([0,1]) \to B_0(M).$$

The mapping $a \to \Lambda_a$ is one-to-one, and for every σ-additive state on M,

$$m(a) = \int_0^1 \lambda m(\Lambda_a(d\lambda)).$$

($f \in \mathcal{T}, h(f) = a,\ \Lambda_a(B) = h \circ f^{-1}(B),\ B \in \mathcal{B}([0,1])$).
(*SP 2005*)

functional calculus

$B_0(M)$ is a Boolean σ-algebra. If $\xi_i, i = 1, 2, \ldots, n$ are sharp real observables on M, there is a unique observable $\eta : \mathcal{B}(\mathbb{R}^n) \to B_0(M)$ such that $\xi_i(A) = \eta(\pi^{-1}(A)$, $\pi : \mathbb{R}^n \to \mathbb{R}$ is the projection. η is the *joint observable* for $\xi_i, i = 1, 2, \ldots, n$.

If $\phi : \mathbb{R}^n \to \mathbb{R}$ is any Borel function, the map $B \to \eta(\phi^{-1}(B))(B \in \mathcal{B}(\mathbb{R}))$ is an observable with range in $B_0(M)$. The observable $\eta \circ \phi^{-1}$ is ϕ-*function of* $\xi_i, i = 1, 2, \ldots, n$.

• For $a, b \in M$, $\Lambda_{a \boxplus b}$ is the ϕ-function of Λ_a, Λ_b where $\phi(t_1, t_2) = \min\{t_1 + t_2, 1\}$,
$\Lambda_{a \vee b} = \phi(\Lambda_a, \Lambda_b), \phi(t_1, t_2) = t_1 \vee t_2$.

smearings

- $(X, \mathcal{T}, h)$ - LS-representation of M,
$(Y, \mathcal{F})$ - measurable space. TFAE:
(i) $\xi : \mathcal{F} \to M$ is an observable.
(ii) $\exists\, \nu : X \times \mathcal{F} \to [0, 1]$:
$\forall F \in \mathcal{F}$, $x \mapsto \nu(x, F) \in \mathcal{T}$:
$(F_i)_i \subseteq \mathcal{F}$, $F_i \cap F_j = \emptyset, i \neq j \implies$
$h(\{x \in X : \sum_{i=1}^{\infty} \nu(x, F_i) \neq \nu(x, \bigcup_{i=1}^{\infty} F_i)\}) = 0.$

$$s(\xi(F)) = \int_X \nu(x, F) s \circ h(dx), s - \sigma\text{-additive}$$

- Every probability measure on the Boolean σ-algebra $B_0(M)$ of sharp elements of M uniquely extends to a σ-additive state on M (*SP 2005*).

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proof

Proof. Put $X = S(E)$, $a \leftrightarrow \hat{a} : S(E) \to [0,1]$, $\hat{a}(s) = s(a)$, continuous function. Define for $f : X \to [0,1] : f \sim \hat{b}, b \in E$ iff $N(f - \hat{b}) \cap ExtS(E)$ is meager. $\mathcal{T}_e$ is the tribe generated by $\{\hat{a} : a \in E\}$, then for all $f \in \mathcal{T}_e$, $\exists! b \in E : \hat{b} \sim f$. Define $h(f) = b$ iff $f \sim \hat{b}$, then $h : \mathcal{T}_e \to E$ is surjective σ-effect algebra morphism.

1. Put $B_0(E)$ - sharp (central) elements in E,
$\forall a \in E, e \in B_0(E) : a = a \wedge e \oplus a \wedge e'$. Then
$a = h(\hat{a}) = h(\widehat{a \wedge e}) + h(\widehat{a \wedge e'})$. Using that for $e \in B_0(E)$,
$\hat{e}(s) = s(e) \in \{0,1\}\ \forall s \in ExtS(E)$, we show that
$h(\widehat{a \wedge e}) = h(\hat{a}) \wedge h(\hat{e})$.

2. E has GC: $a, b \in E$, $f_a, f_b \in \mathcal{T}_e$, $h(f_a) = a, h(f_b) = b$.
$\mathcal{T}_e$ has GC $\implies \exists\, A \in \mathcal{B}_0(\mathcal{T}_e)$ such that

$$f_a \wedge \chi_A \leq f_b \wedge \chi_A, \quad f_a \wedge \chi_{A^c} \geq f_b \geq \chi_{A^c}$$

observables

An *observable* on a σ MV-(effect) algebra M is a σ-morphism $\xi : \mathcal{F} \to M$, where $(Y, \mathcal{F})$ is a measurable space, that is:

(i) $\xi(Y) = 1$;

(ii) $\xi(A \cup B) = \xi(A) \oplus \xi(B)$ whenever $A, B \in \mathcal{F}, A \cap B = \emptyset$,

(iii) $A, A_n \in \mathcal{F}$, $A_n \nearrow A$ implies $\xi(A_n) \nearrow \xi(A)$.

ξ is *sharp* if its range is contained in the set $B_0(M)$ of sharp elements in M.

ξ is *real* if $(Y, \mathcal{F}) \equiv (\mathbb{R}, \mathcal{B}(\mathbb{R}))$; the *expectation* of ξ in a state m is $m(\xi) = \int_{\mathbb{R}} \lambda m \circ \xi(d\lambda)$.