Quasi-equational theories of relational lattices[1]

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Real world computer science

Relational lattices

Quasiequational theories of relational lattices

The lattice of a frame

p-morphisms from lattice embeddings

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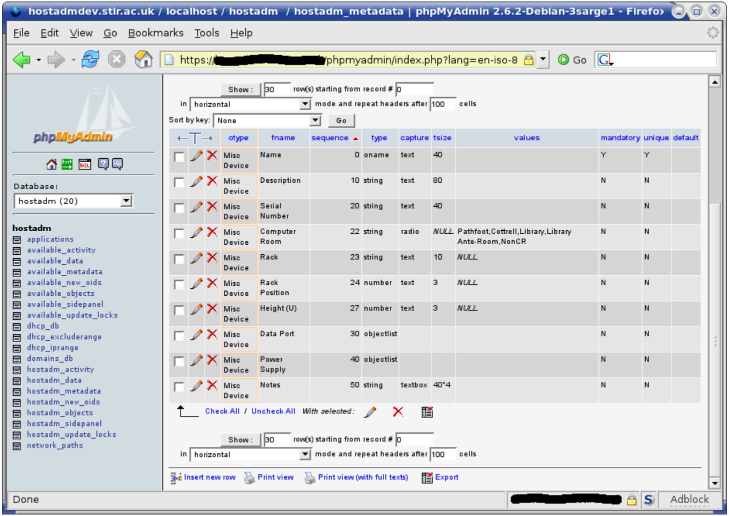
Databases, tables, sqls . . .



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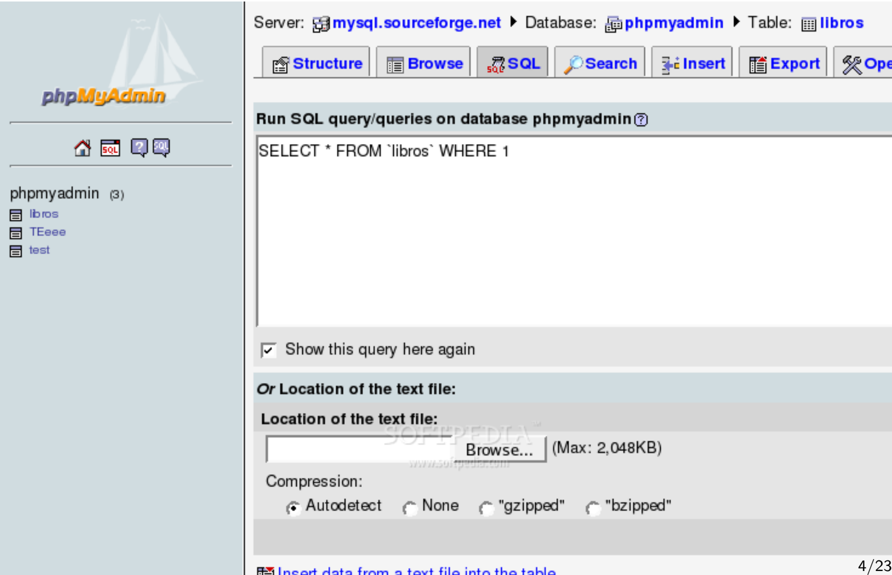

Operations on tables: the natural join

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\bowtie$

Item	Description
33	Book
33	Livre
21	Machine

$=$

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Operations on tables: the inner union

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\cup$

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

$=$

Name	Surname
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Lattices from databases

Proposition. [Spight & Tropashko, 2006] The set of tables, whose columns are indexed by a subset of A and values are from a set D, is a lattice, with natural join as meet and inner union as join.

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The relational lattices $\mathsf{R}(D, A)$

A a set of attributes, D a set of values.

An element of $\mathsf{R}(D, A)$:

- a pair (X, T) with $X \subseteq A$ and $T \subseteq D^X$.

We have

$$(X_1, T_1) \leq (X_2, T_2) \text{ iff } X_2 \subseteq X_1 \text{ and } T_1\upharpoonright_{X_2} \subseteq T_2\,.$$

NB :
this is the Grothendieck construction of a contravariant functor.

Meet and join

$$
\begin{aligned}
(X_1, T_1) \wedge (X_2, T_2) &:= (X_1 \cup X_2, T) \\
\text{where } T &= \{\, f \mid f_{\restriction X_i} \in T_i, i = 1, 2 \,\} \\
&= i_{X_1 \cup X_2}(T_1) \cap i_{X_1 \cup X_2}(T_2)\,,
\end{aligned}
$$

$$
\begin{aligned}
(X_1, T_1) \vee (X_2, T_2) &:= (X_1 \cap X_2, T) \\
\text{where } T &= \{\, f \mid \exists i \in \{\, 1, 2 \,\}, \exists g \in T_i \text{ s.t. } g_{\restriction X_1 \cap X_2} = f \,\} \\
&= T_1 {\upharpoonright\!\!\upharpoonright}_{X_1 \cap X_2} \cup T_2 {\upharpoonright\!\!\upharpoonright}_{X_1 \cap X_2} \enspace .
\end{aligned}
$$

Representation via closure operators

The Hamming/Priess_Crampe-Ribenboim ultrametric distance on D^A:

$$\delta(f, g) := \{\, x \in A \mid f(x) \neq g(x) \,\}\,.$$

NB: this distance takes values in the join-semilattice $(P(A), \emptyset, \cup)$.

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A subset X of $A + D^A$ is *closed* if $\delta(f, g) \cup \{ g \} \subseteq X$ implies $f \in X$.

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Proposition. [Litak, Mikulás and Hidders 2015] $\mathrm{R}(D, A)$ is isomorphic to the lattice of closed subsets of $A + D^A$.

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Undecidable quasiequational theories

Theorem (Litak, Mikulás and Hidders, 2015)
The set of quasiequations in the signature $(\wedge, \vee, H)$ that are valid on relational lattices is undecidable.

We refine here this to:

Theorem
The set of quasiequations in the signature $(\wedge, \vee)$ that are valid on relational lattices is undecidable.

We actually prove a stronger result:

Theorem
It is undecidable whether a finite subdirectly irreducible lattice embeds into some $\mathrm{R}(D, A)$.

Related undecidable problems

Theorem (Maddux)

The equational theory of 3*-dimensional diagonal free cylindric algebras is undecidable.*

Theorem (Hirsch and Hodkinson)

It is not decidable whether a finite simple relation algebras embeds into a concrete one (a powerset of a binary product).

Theorem (Hirsch, Hodkinson and Kurucz)

It is not decidable whether a finite mutimodal frame has a surjective p-morphism from a universal product frame.

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The lattice of a frame

Let $\mathcal{F} = (X, \{ R_a \mid a \in A \})$ be a finite A-frame.

If $\alpha \subseteq A$, then we say that $Y \subseteq X$ is α-closed if

$$x_0 R_{a_1} x_1 R_{a_2} x_2 \ldots R_{a_n} x_n \in Y \text{ and } \{ a_1, \ldots, a_n \} \subseteq \alpha$$
$$\text{implies } x_0 \in Y\,.$$

We say that $Z \subseteq A + X$ is closed if $Z \cap X$ is $Z \cap A$-closed.

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We say that $Z \subseteq A + X$ is closed if $Z \cap X$ is $Z \cap A$-closed.

Definition. The lattice $\mathsf{L}(\mathcal{F})$ is the lattice of closed subsets of $A + X$.

We prove:

Theorem
A full rooted $\mathcal{F}$ has a surjective p-morphism from a universal product frame iff $\mathsf{L}(\mathcal{F})$ embeds into a relational lattice.

The easy part: embeddings from p-morphisms

L extends to a contravariant functor.

Moreover if $X = \prod_{a \in A} D$ $(= D^A)$ and A is finite then $\mathrm{L}(\mathcal{F}) = \mathrm{R}(D, A)$.

The easy part: embeddings from p-morphisms

L extends to a contravariant functor.

Moreover if $X = \prod_{a \in A} D$ ($= D^A$) and A is finite then $\mathsf{L}(\mathcal{F}) = \mathsf{R}(D, A)$.

Corollary. If a finite multimodal frame $\mathcal{F}$ has a p-morphism from a universal product frame, then $L(\mathcal{F})$ embeds into some $\mathsf{R}(D, A)$.

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Lattice embeddings into the R(D, A)s

We study lattice embeddings of the form

$$i : L \longrightarrow \mathrm{R}(D, A)$$

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1. i preserves $\bot, \top$, so $\mu \dashv i$ (use L subdirectly irreducible);
2. $A \subseteq J(L)$ is the set of join-prime elements of L.

Then

$$v(f) := \{ j \in A \mid j \leq \mu(f) \}$$

is a "module" on the space (D^A, δ).

The subspace induced

$$F_0 = \{\, f \in D^A \mid \nu(f) = \emptyset \,\} = \{\, f \in D^A \mid \nu(f) \in J(L) \setminus A \,\}$$

is the kernel of a module, therefore it is "pairwise-complete".

Some theory of (generalized) ultrametric spaces over $P(A)$

The subspace induced

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Theorem

Injective objects (pairwise-complete and spherically complete) in the category of GUM over $P(A)$ are spaces of sections (dependent product types, Hamming graphs, universal product frames, . . .)

Completing the proof of the converse

Moral: when A is finite, up to isomorphism, pairwise-complete complete subspace are universal product frames.

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When $L = L(\mathcal{F})$, the restriction of μ to F_0 yields the desired p-morphism.

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When $L = L(\mathcal{F})$, the restriction of μ to F_0 yields the desired p-morphism.

For this, we also need the following Lemma, proved in our previous work on axiomatisations of relational lattices.

Lemma

If L is a finite atomistic lattice in the variety generated by relational lattices, then every non-trivial minimal (irredundant) join-cover has exactly one element which is not join-prime.

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