

# Quasi-equational theories of relational lattices<sup>1</sup>

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<sup>1</sup>Preprint available on HAL:

# Plan

Real world computer science

Relational lattices

Quasiequational theories of relational lattices

The lattice of a frame

$\rho$ -morphisms from lattice embeddings

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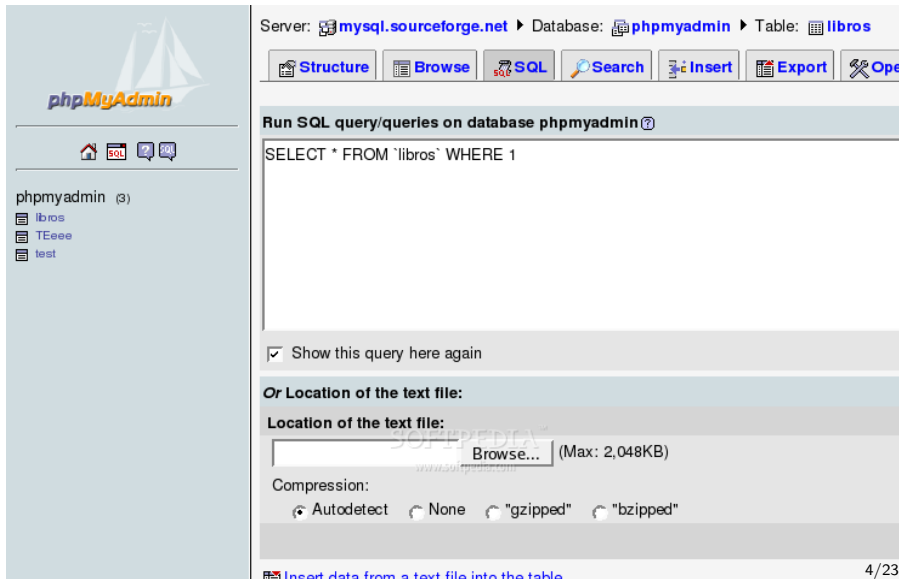
$\rho$ -morphisms from lattice embeddings

# Databases, tables, sqls ...

The screenshot shows the phpMyAdmin interface in a browser window. The browser address bar shows the URL: `https://[redacted]phpmyadmin/index.php?lang=en-iso-8`. The interface includes a menu bar (File, Edit, View, Go, Bookmarks, Tools, Help), a search bar, and a sidebar with the phpMyAdmin logo and a list of databases for the 'hostadm' server. The main area displays the structure of a table, with columns: `otype`, `fname`, `sequence`, `type`, `capture`, `tsize`, `values`, `mandatory`, `unique`, and `default`. The table contains 10 rows of data, each representing a field in the table. The 'otype' column indicates the field type (e.g., Misc Device), and the 'values' column shows the field's value (e.g., NULL, Pathfoot, Cottrell, Library, Library Ante-Room, NonCR).

otype	fname	sequence	type	capture	tsize	values	mandatory	unique	default
Misc Device	Name	0	oname	text	40		Y	Y	
Misc Device	Description	10	string	text	80		N	N	
Misc Device	Serial Number	20	string	text	40		N	N	
Misc Device	Computer Room	22	string	radio	NULL	Pathfoot,Cottrell,Library,Library Ante-Room,NonCR	N	N	
Misc Device	Rack	23	string	text	10	NULL	N	N	
Misc Device	Rack Position	24	number	text	3	NULL	N	N	
Misc Device	Height (U)	27	number	text	3	NULL	N	N	
Misc Device	Data Port	30	objectlist				N	N	
Misc Device	Power Supply	40	objectlist				N	N	
Misc Device	Notes	50	string	textbox	40*4		N	N	

# Databases, tables, sqls ...



The screenshot displays the phpMyAdmin web interface. At the top, the server is identified as `mysql.sourceforge.net`, the database as `phpmyadmin`, and the table as `libros`. A navigation bar includes buttons for Structure, Browse, SQL, Search, Insert, Export, and Open. The main area is titled "Run SQL query/queries on database phpmyadmin". The SQL query entered is `SELECT * FROM `libros` WHERE 1`. Below the query, there is a checkbox labeled "Show this query here again" which is checked. A section titled "Or Location of the text file:" contains a "Location of the text file:" label, a text input field, a "Browse..." button, and a "(Max: 2,048KB)" limit. Underneath, the "Compression:" section has radio buttons for "Autodetect", "None", "gzipped", and "bzipped". The bottom of the page shows a partially visible button for "Insert data from a text file into the table".

Server: `mysql.sourceforge.net` ▶ Database: `phpmyadmin` ▶ Table: `libros`

Structure Browse SQL Search Insert Export Open

Run SQL query/queries on database phpmyadmin ?

```
SELECT * FROM `libros` WHERE 1
```

Show this query here again

Or Location of the text file:

Location of the text file:

Browse... (Max: 2,048KB)

Compression:

Autodetect  None  "gzipped"  "bzipped"

Insert data from a text file into the table

## Operations on tables: the natural join

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

 $\bowtie$ 

Item	Description
33	Book
33	Livre
21	Machine

=

Name	Surname	Item	Description
Luigi	Santocanale	33	Book
Luigi	Santocanale	33	Livre
Alan	Turing	21	Machine

## Operations on tables: the inner union

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

U

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

=

Name	Surname
Luigi	Santocanale
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Usain	Bolt

# Lattices from databases

**Proposition.** [Spight & Tropashko, 2006] The set of tables, whose columns are indexed by a subset of  $A$  and values are from a set  $D$ , is a lattice, with natural join as meet and inner union as join.



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# The relational lattices $R(D, A)$

$A$  a set of attributes,  $D$  a set of values.

An element of  $R(D, A)$ :

- ▶ a pair  $(X, T)$  with  $X \subseteq A$  and  $T \subseteq D^X$ .

We have

$$(X_1, T_1) \leq (X_2, T_2) \text{ iff } X_2 \subseteq X_1 \text{ and } T_1 \upharpoonright_{X_2} \subseteq T_2.$$

NB :

this is the Grothendieck construction of a contravariant functor.

## Meet and join

$$(X_1, T_1) \wedge (X_2, T_2) := (X_1 \cup X_2, T)$$

where  $T = \{f \mid f \upharpoonright_{X_i} \in T_i, i = 1, 2\}$   
 $= i_{X_1 \cup X_2}(T_1) \cap i_{X_1 \cup X_2}(T_2),$

$$(X_1, T_1) \vee (X_2, T_2) := (X_1 \cap X_2, T)$$

where  $T = \{f \mid \exists i \in \{1, 2\}, \exists g \in T_i \text{ s.t. } g \upharpoonright_{X_1 \cap X_2} = f\}$   
 $= T_1 \upharpoonright_{X_1 \cap X_2} \cup T_2 \upharpoonright_{X_1 \cap X_2} .$

## Representation via closure operators

The Hamming/Priess-Crampe-Ribenboim ultrametric distance on  $D^A$ :

$$\delta(f, g) := \{x \in A \mid f(x) \neq g(x)\}.$$

NB: this distance takes values in the join-semilattice  $(P(A), \emptyset, \cup)$ .

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A subset  $X$  of  $A + D^A$  is *closed* if  $\delta(f, g) \cup \{g\} \subseteq X$  implies  $f \in X$ .

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**Proposition.** [Litak, Mikulás and Hidders 2015]  $R(D, A)$  is isomorphic to the lattice of closed subsets of  $A + D^A$ .

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# Undecidable quasiequational theories

Theorem (Litak, Mikulás and Hidders, 2015)

*The set of quasiequations in the signature  $(\wedge, \vee, H)$  that are valid on relational lattices is undecidable.*

We refine here this to:

Theorem

*The set of quasiequations in the signature  $(\wedge, \vee)$  that are valid on relational lattices is undecidable.*

We actually prove a stronger result:

Theorem

*It is undecidable whether a finite subdirectly irreducible lattice embeds into some  $R(D, A)$ .*



## Related undecidable problems

### Theorem (Maddux)

*The equational theory of 3-dimensional diagonal free cylindric algebras is undecidable.*

### Theorem (Hirsch and Hodkinson)

*It is not decidable whether a finite simple relation algebras embeds into a concrete one (a powerset of a binary product).*

### Theorem (Hirsch, Hodkinson and Kurucz)

*It is not decidable whether a finite multimodal frame has a surjective  $p$ -morphism from a universal product frame.*

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## The lattice of a frame

Let  $\mathcal{F} = (X, \{R_a \mid a \in A\})$  be a finite  $A$ -frame.

If  $\alpha \subseteq A$ , then we say that  $Y \subseteq X$  is  $\alpha$ -closed if

$$x_0 R_{a_1} x_1 R_{a_2} x_2 \dots R_{a_n} x_n \in Y \text{ and } \{a_1, \dots, a_n\} \subseteq \alpha \\ \text{implies } x_0 \in Y.$$

We say that  $Z \subseteq A + X$  is closed if  $Z \cap X$  is  $Z \cap A$ -closed.

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**Definition.** The lattice  $L(\mathcal{F})$  is the lattice of closed subsets of  $A + X$ .

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**Definition.** The lattice  $L(\mathcal{F})$  is the lattice of closed subsets of  $A + X$ .

We prove:

## Theorem

*A full rooted  $\mathcal{F}$  has a surjective  $p$ -morphism from a universal product frame iff  $L(\mathcal{F})$  embeds into a relational lattice.*

## The easy part: embeddings from $p$ -morphisms

$L$  extends to a contravariant functor.

Moreover if  $X = \prod_{a \in A} D (= D^A)$  and  $A$  is finite then  $L(\mathcal{F}) = R(D, A)$ .

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Moreover if  $X = \prod_{a \in A} D (= D^A)$  and  $A$  is finite then  $L(\mathcal{F}) = R(D, A)$ .

**Corollary.** If a finite multimodal frame  $\mathcal{F}$  has a  $p$ -morphism from a universal product frame, then  $L(\mathcal{F})$  embeds into some  $R(D, A)$ .

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## Lattice embeddings into the $R(D, A)$ s

We study lattice embeddings of the form

$$i : L \rightarrow R(D, A)$$

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We can suppose that:

1.  $i$  preserves  $\perp, \top$ , so  $\mu \dashv i$  (use  $L$  subdirectly irreducible);
2.  $A \subseteq J(L)$  is the set of join-prime elements of  $L$ .

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Then

$$v(f) := \{j \in A \mid j \leq \mu(f)\}$$

is a “module” on the space  $(D^A, \delta)$ .

# Some theory of (generalized) ultrametric spaces over $P(A)$

The subspace induced

$$F_0 = \{f \in D^A \mid \nu(f) = \emptyset\} = \{f \in D^A \mid \nu(f) \in J(L) \setminus A\}$$

is the kernel of a module, therefore it is “pairwise-complete”.

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is the kernel of a module, therefore it is “pairwise-complete”.

## Theorem

*Injective objects (pairwise-complete and spherically complete) in the category of GUM over  $P(A)$  are spaces of sections (dependent product types, Hamming graphs, universal product frames, ...)*

## Completing the proof of the converse

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When  $L = L(\mathcal{F})$ , the restriction of  $\mu$  to  $F_0$  yields the desired  $p$ -morphism.

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Moral: when  $A$  is finite, up to isomorphism, pairwise-complete complete subspace are universal product frames.

When  $L = L(\mathcal{F})$ , the restriction of  $\mu$  to  $F_0$  yields the desired  $\rho$ -morphism.

For this, we also need the following Lemma, proved in our previous work on axiomatisations of relational lattices.

### Lemma

*If  $L$  is a finite atomistic lattice in the variety generated by relational lattices, then every non-trivial minimal (irredundant) join-cover has exactly one element which is not join-prime.*



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