On the relational degree of a clone

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The relational degree of an algebra

Definition

Let $\mathbf{A}$ be an algebra.

$$\deg(\mathbf{A}) = \min\{d \in \mathbb{N} \,|\, \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathrm{Sub}(\mathbf{A}^d))\}.$$

If no such d exists, $\mathbf{A}$ is *not finitely related*.

Hence $\deg(\mathbf{A}) = \min\{d \in \mathbb{N} \,|\, \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}\,\mathrm{Inv}^{[d]}\,\mathrm{Clo}(\mathbf{A})\}$.

Relational degree and interpolation

Definition

Let C be a clone on A, $d \in \mathbb{N}$. Then

$$\mathrm{Loc}_d(C) = \{ l : A^n \to A \,|\, n \in \mathbb{N}, \\ \forall S \subseteq A^n : |S| \le d \Rightarrow \\ \exists c \in C : c|_S = l|_S \}.$$

Theorem (Eigenthaler?)

Let C be a clone on A, $d \in \mathbb{N}$. Then

- $\mathrm{Loc}_d(C) = \mathrm{Pol}\,\mathrm{Inv}^{[d]}(C)$.
- $\deg(\mathbf{A}) = \min\{ d \in \mathbb{N} \,|\, \mathrm{Loc}_d(C) \subseteq C \}$ if such a d exists.

Finitely related algebras

Examples

The following algebras are finitely related:

1. finite algebras with Mal'cev term.
2. finite algebras with edge term (Aichinger, Mayr, McKenzie, 2009).
3. commutative semigroups. (Dayey, Jackson, Pitkethly, Szabó, 2011)

Examples

The following algebras are not finitely related:

1. $(\{0,1\}, x \wedge (y \vee z))$. (Post & classic clone theory)
2. $(\{0,1\}, \rightarrow)$.
3. The 6-element Brandt monoid (seen as a semigroup) (Mayr, 2013).

Proof techniques

How to prove "**A** is finitely related"

1. Come up with a set of relations R such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(R)$.
2. Look for completeness results: **A** might be primal, endoprimal, quasiprimal.
3. **A** might be affine complete, polynomially rich.
4. For finite **A**: Is there an infinite chain of clones intersecting to $\mathrm{Clo}(\mathbf{A})$?

Proof techniques

How to prove "**A** is not finitely related"

1. For finite **A**: Find an infinite chain of clones intersecting to Clo(**A**)!
2. For every n, find a function that can be interpolated at all choices of n places by a term function, but is not a term function.
3. For finite **A**: Find infinitely many n such that there is n-ary f with $f \notin \mathrm{Clo}(\mathbf{A})$ and all $n-1$-ary minors are in Clo(**A**).

Non-preservation results

There is a three element groupoid $\mathbf{A} = (A, *)$ such that

1. $\mathbf{A}$ is finitely related, but it has $\mathbf{I} = (\{0, 1\}, \to)$ as subalgebra and homomorphic image.
2. $\mathbf{I}_{0,1} = (\{0, 1\}, \to, 0, 1)$ and $\mathbf{I}_{1,0} = (\{0, 1\}, \to, 1, 0)$ are finitely related, but $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ is not.

Preservation results

Theorem (Markovic, Maroti, McKenzie)

Let $\mathbf{R}, \mathbf{C}$ be finite algebras such that $\mathbf{R} \leq \mathbf{C}^n$ and $\pi_1(R) = C$. Then $\mathbf{R}$ is finitely related $\Leftrightarrow$ $\mathbf{C}$ is finitely related.

Corollary (Davey, Jackson, Pitkethly, Szabo)

Let $\mathbf{A}, \mathbf{B}$ be finite algebras such that $V(\mathbf{A}) = V(\mathbf{B})$. If $\mathbf{A}$ is finitely related, then so is $\mathbf{B}$.

Remark

In this case Clo($\mathbf{A}$) and Clo($\mathbf{B}$) are isomorphic as abstract clones.

A preservation result

Theorem

Let $\mathbf{A}$ be an algebra of relational degree m, let I be a set, and for each $i \in I$, let $\mathbf{B}_i$ be an algebra in $\mathbb{SH}(\mathbf{A})$. Let $\mathbf{C} \leq \mathbf{A} \times \prod_{i \in I} \mathbf{B}_i$. We assume that π_A is surjective, i.e., $\pi_A(C) = A$. Then $\mathbf{C}$ is of relational degree at most $\max(2, m)$.

Lemma

Let $k, m \in \mathbb{N}$, and let $\mathbf{A}$ be a finite k-generated algebra. If $\mathbf{F}_{V(\mathbf{A})}(k)$ is of relational degree m, then $\mathbf{A}$ is of relational degree at most $m \cdot |A|^k$.

Theorem

Let $\mathbf{A}$ be a finite algebra of finite type with edge term such that $V(\mathbf{A})$ is residually small. Then there is $k \in \mathbb{N}$ such that every algebra in $\mathbb{V}(\mathbf{A})$ is of relational degree at most k.

Clones on infinite sets

Let C be a clone on A. $\mathrm{Inv}(C) = \bigcup_{n\in\mathbb{N}} \mathrm{Inv}^{[n]}(C)$.

Definition

C is *locally closed* $:\Leftrightarrow C = \mathrm{Pol}\,\mathrm{Inv}\,C$.

Theorem

Let A be a set with $|A| = \aleph_0$, and let C be a clone with quasigroup operations. If $|\mathrm{Pol}\,\mathrm{Inv}\,C| \leq \aleph_0$, then $C = \mathrm{Pol}\,\mathrm{Inv}\,C$.

Theorem

There exist a set A with $|A| = \aleph_0$ and a constantive clone C on A such that $|\mathrm{Pol}\,\mathrm{Inv}\,C| = \aleph_0$ and $C \neq \mathrm{Pol}\,\mathrm{Inv}\,C$.

Theorem

Let A be a set with $|A| = \aleph_0$, and let C be a clone on A with quasigroup operations. Then the following are equivalent:

1. $|\mathrm{Pol}\,\mathrm{Inv}\,C| \leq \aleph_0$.
2. For each $n \in \mathbb{N}$, $C^{[n]}$ has a finite base of equality.
3. $|C| \leq \aleph_0$ and $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} : C^{[n]} = (\mathrm{Loc}_k C)^{[n]}$.
4. $|C| \leq \aleph_0$ and $C = \mathrm{Pol}\,\mathrm{Inv}\,C$.