Q-sup-lattices and Q-sup-algebras

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Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be posets. Monotone mappings $f\colon X \to Y$ and $g\colon Y \to X$ are *adjoint* if

$$f(x) \leq_Y y \iff x \leq_X g(y)$$

for any $x \in X$, $y \in Y$.

f 'left adjoint', 'residuated', preserves existing joins
g 'right adjoint', 'residual', preserves existing meets

Sup-lattice – complete join-semilattice (actually a complete lattice)

Homomorphism – join-preserving mapping

$$f\left(\bigvee X\right) = \bigvee\{f(x) \mid x \in X\}$$

Category **Sup**

Quantale – a sup-lattice Q with an associative binary operation '$\cdot$' satisfying $q \cdot (\bigvee A) = \bigvee\{q \cdot a \mid a \in A\}$ and $(\bigvee A) \cdot q = \bigvee\{a \cdot q \mid a \in A\}$

Unital if Q has a multiplicative unit e

Commutative if $\cdot$ is commutative

Homomorphism $f(\bigvee A) = \bigvee\{f(a) \mid a \in A\}$ and $f(q \cdot r) = f(q) \cdot f(r)$

Category **Quant** (**UnQuant**) – semigroups (monoids) in **Sup**

Left Q-module – a sup-lattice M with left action $*$ of the quantale satisfying $q * (\bigvee N) = \bigvee\{q * n \mid n \in N\}$,
$(\bigvee R) * m = \bigvee\{r * m \mid r \in R\}$,
$(q \cdot r) * m = q * (r * m)$.

Unital if Q is unital and $me = m$ for all m

Homomorphism $f(\bigvee m_i) = \bigvee f(m_i)$, $f(mq) = f(m)q$

Category Q-**Mod**

If Q is unital, Q-modules are also assumed as unital

Let Q be a commutative quantale

Q-algebra – a left Q-module A with an associative binary operation '$\cdot$' such that A is a quantale, and both $a \cdot -$ and $- \cdot a$ are module homomorphisms for any $a \in A$ $(a \cdot (q * b) = q * (a \cdot b) = (q * a) \cdot b)$

Homomorphism – Q-module homomorphism that also preserves multiplication

Category Q-**Alg**

Analogy: abelian groups, rings, modules, algebras

For any $q \in Q$, both mappings $q \cdot -$ and $- \cdot q$ preserve joins $\implies$ they have right adjoints $q \to -$ and $- \leftarrow q$ characterized by
$r \leq q \to s \iff q \cdot r \leq s \iff q \leq s \leftarrow r$

If Q is commutative, $\to$ and $\leftarrow$ coincide

Explicitly, $q \to s = \bigvee\{r \in Q \mid q \cdot r \leq s\}$

$$q \to \bigwedge S = \bigwedge_{s \in S}(q \to s) \qquad \bigvee S \to r = \bigwedge_{s \in S}(s \to r)$$

$$\bot \to r = \top \qquad q \to (r \to s) = (q \cdot r) \to s$$

If Q is unital:

$$1 \to r = r \qquad q \to q \geq 1$$

From now on, assume that Q is a fixed commutative unital quantale (the 'base quantale').

Definition

Let X be a set. A mapping $e\colon X \times X \to Q$ is called a *Q-order* if for any $x, y, z \in X$ the following are satisfied:

1. $e(x, x) \geq 1$ (reflexivity),
2. $e(x, y) \cdot e(y, z) \leq e(x, z)$ (transitivity),
3. if $e(x, y) \geq 1$ and $e(y, x) \geq 1$, then $x = y$ (antisymmetry).

The pair (X, e) is then called a *Q-ordered set*.

$x = y$ iff $e(x, z) = e(y, z)$ for any $z \in X$ iff $e(z, x) = e(z, y)$ for any $z \in X$.

For a Q-order e on X, the relation $\leq_e$ defined as $x \leq_e y \iff e(x,y) \geq 1$ is a partial order in the usual sense (the *induced partial order*).

This means that any Q-ordered set can be viewed as an ordinary poset satisfying additional properties.

Vice versa, for a partial order $\leq$ on a set X and any quantale Q we can define a Q-order $e_{\leq}$ by

$$e_{\leq}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \bot & \text{otherwise.} \end{cases}$$

Example

1. The base quantale Q itself can be regarded as a Q-ordered set via $e(q,r) = q \to r$ as for any $q, r, s \in Q$ we have:
   1. $1 \cdot q = q$ implies $1 \leq q \to q$,
   2. $q \cdot (q \to r) \cdot (r \to s) \leq r \cdot (r \to s) \leq s$, which is equivalent to $(q \to r) \cdot (r \to s) \leq q \to s$,
   3. $1 \leq q \to r$ iff $q = 1 \cdot q \leq r$, similarly $1 \leq q \to q$ iff $r \leq q$.
2. The same actually works for any Q-module L when putting $e(l, m) = l \to_Q m = \bigvee\{q \in Q \mid q * l \leq m\}$.

Definition

Let (X, e_X), (Y, e_Y) be Q-ordered sets. A mapping $f\colon X \to Y$ is called *Q-monotone* if $e_X(x, y) \leq e_Y(f(x), f(y))$ for any $x, y \in X$.

Like with the case of Q-orders, any Q-monotone mapping $(X, e_X) \to (Y, e_Y)$ is an ordinary monotone mapping $(X, \leq_{e_X}) \to (Y, \leq_{e_Y})$ with respect to the induced partial orders.

Conversely, a monotone mapping $(X, \leq_X) \to (Y, \leq_Y)$ becomes a Q-monotone mapping $(X, e_{\leq_X}) \to (Y, e_{\leq_Y})$ with respect to the Q-orders induced by $\leq_X$ and $\leq_Y$.

Definition

Let $f\colon (X, e_X) \to (Y, e_Y)$, $g\colon (Y, e_Y) \to (X, e_X)$ be Q-monotone mappings.
We say that (f, g) is a *Q-adjunction*, if $e_Y(f(x), y) = e_X(x, g(y))$ for any $x \in X$, $y \in Y$.
Then f is called a *left* and g a *right Q-adjoint*.

Again, a Q-adjunction is an ordinary adjunction satisfying an additional property.

Definition

A *Q-subset* of a set X is an element of the set Q^X.

This corresponds to viewing ordinary subsets of X as elements of $\mathbf{2}^X$ where $\mathbf{2}$ is the two-element quantale.

Definition

Let M be a Q-subset of a Q-ordered set (X, e). An element s of X is called a *Q-join* of M, denoted $\bigsqcup M$ if:

1. $M(x) \leq e(x, s)$ for all $x \in X$, and
2. for all $y \in X$, $\bigwedge_{x \in X}(M(x) \to e(x, y)) \leq e(s, y)$.

By analogy, an element m of X is called a *Q-meet* of M, denoted $\bigsqcap M$, if:

1. $M(x) \leq e(m, x)$ for all $x \in X$, and
2. for all $y \in X$, $\bigwedge_{x \in X}(M(x) \to e(y, x)) \leq e(y, m)$.

Proposition

Let (X, e) be a Q-ordered set, $s, m \in X$ and $M \in Q^X$.

1. $s = \bigsqcup M$ *iff for all* $y \in X$, $e(s, y) = \bigwedge_{x \in X}(M(x) \to e(x, y))$,
2. $m = \bigsqcap M$ *iff for all* $y \in X$, $e(y, m) = \bigwedge_{x \in X}(M(x) \to e(y, x))$.

The antisymmetry property of the Q-order implies that if a Q-join (Q-meet) of a Q-subset exists, it is unique.

Definition

A Q-ordered set (X, e) is called:

1. *Q-join-complete* if $\bigsqcup M$ exists for any Q-subset M of X,
2. *Q-meet-complete* if $\bigsqcap M$ exists for any Q-subset M of X,
3. *Q-complete* if it is both Q-join-complete and Q-meet-complete.

Proposition

Let (X, e) be a Q-complete Q-ordered set. Then $(X, \leq_e)$ is a complete poset, and for any $S \subseteq X$ we have $\bigvee S = \bigsqcup \varphi_S$ where

$$\varphi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ \bot & \text{otherwise.} \end{cases}$$

We will denote a Q-complete set (X, e) by $(X, \bigsqcup)$.

Definition

Let X and Y be sets, and $f : X \to Y$ be mapping. We define *Zadeh's forward power set operator* that maps Q-subsets of X to Q-subsets of Y as:

$$f_Q^{\to}(M)(y) = \bigvee_{x \in f^{-1}(y)} M(x).$$

Definition

Let (X, e_X) and (Y, e_Y) be Q-ordered sets. We say that a mapping $f : X \to Y$ is *Q-join-preserving* if for any Q-subset M of X such that $\bigsqcup_X M$ exists, $\bigsqcup_Y f_Q^{\to}(M)$ exists and

$$f\left(\bigsqcup\nolimits_X M\right) = \bigsqcup\nolimits_Y f_Q^{\to}(M).$$

Proposition

Let (X, e_X), (Y, e_Y) be Q-ordered sets, and $f: X \to Y$, $g: Y \to X$ be two mappings. Then the following hold:

1. *If (X, e_X) is Q-complete, then f is Q-monotone and has a right adjoint iff $f\left(\bigsqcup_X M\right) = \bigsqcup_Y f_Q^{\to}(M)$ for all $M \in Q^X$.*
2. *If (Y, e_Y) is Q-complete, then g is Q-monotone and has a left adjoint iff $g\left(\bigsqcap_Y N\right) = \bigsqcap_X g_Q^{\to}(N)$ for all $N \in Q^Y$.*

Definition

A *Q-sup-lattice* is a Q-join-complete set. *Homomorphisms of Q-sup-lattices* are Q-join-preserving mappings. By Q-**Sup** we will denote the category whose objects are Q-sup-lattices, and morphisms are Q-sup-lattice homomorphisms.

Let I be a set, $(X_i, \bigsqcup_{X_i})$ be Q-sup-lattices for all $i \in I$, and M be a Q-subset of the set $\prod_{i \in I} X_i$. For each $j \in I$ and $y \in X_j$ we then define $\widehat{M}_j \in Q^{X_j}$ as

$$\widehat{M}_j(y) = \bigvee \left\{ M((x_i)_{i \in I}) \mid (x_i)_{i \in I} \in \prod_{i \in I} X_i, \ x_j = y \right\}.$$

Proposition

Let $(X_i, \bigsqcup_{X_i})$ be Q-sup-lattices for all $i \in I$. Then $\underline{X} = \prod_{i \in I} X_i$ is a Q-sup-lattice too, with $\bigsqcup_{\underline{X}} M = (\bigsqcup_{X_i} \widehat{M}_i)_{i \in I}$.

Proposition

Let $(X_i, \bigsqcup_{X_i})$, $i \in I$, be Q-sup-lattices. Then their product is $(\prod_{i \in I} X_i, \bigsqcup_{\underline{X}})$.

Example

For any set X, the set of its Q-subsets (Q^X, sub_X) is Q-complete: For $M \in Q^{Q^X}$ put $S \in Q^X$ as $S(x) = \bigvee_{P \in Q^X} M(P) \cdot P(x)$. Applying the characterization of Q-joins, we can verify that S is the Q-join of M.

Proposition

*The category Q-**Sup** has equalizers: Let $f, g\colon X \to Y$ be two homomorphisms. Then their equalizer is $((X_{fg}, \bigsqcup_X|_{X_{fg}}), \varepsilon)$ where $X_{fg} = \{x \in X | f(x) = g(x)\}$ and $\varepsilon\colon X_{fg} \to X$ is the set inclusion.*

Corollary

*The category Q-**Sup** is complete.*

Example

The terminal object in the category Q-**Sup** is the one-element Q-sup-lattice $(\{*\}, e_*)$ with $e_*(*, *) = \top$.

Proposition

*The category Q-**Sup** has coproducts.*

Proposition

*The category Q-**Sup** has coequalizers.*

Both are constructed using products resp. equalizers, adjoints, and dual Q-orders ($e'(x, y) = e(y, x)$).

Proposition

For Q-sup-lattices X and Y, the set $\mathrm{Hom}(X, Y)$ is also a Q-sup-lattice.

Example

A *Q-quantale* is a Q-sup-lattice A endowed with an associative binary operation $\cdot$ which is Q-join-preserving in both components, namely for any $a \in A$ and $M \in Q^A$,

$$\left(\bigsqcup M\right) \cdot a = \bigsqcup (- \cdot a)^{\rightarrow}_{Q}(M)$$

$$a \cdot \left(\bigsqcup M\right) = \bigsqcup (a \cdot -)^{\rightarrow}_{Q}(M)$$

(Q-quantales are semigroup objects in Q-**Sup** like quantales are semigroup objects in **Sup**.)

Example (Cont.)

Typical example: a commutative quantale Q with $e(x, y) = x \to y$

A new instance: Let X be a Q-sup-lattice, then the set of all its endomorphisms $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ is a Q-quantale with composition of mappings as the multiplication.

(One can verify that $\bigsqcup (g \circ -)^{\rightarrow}_{Q}(M) = g \circ \bigsqcup M$ for any $M \in Q^{\mathrm{End}(X)}$.)

Definition

Let (X, e_X) be a Q-ordered set.

- A mapping $f: X \to X$ is *Q-inflating* if $e_X(x, f(x)) \geq 1$ for any $x \in X$.
- A Q-monotone, Q-inflating mapping is called a *Q-order prenucleus*.
- An idempotent Q-order prenucleus is called a *Q-order nucleus*.

Definition

Let (X, e_X) be a Q-ordered set.

- A mapping $f\colon X \to X$ is *Q-deflating* if $e_X(f(x), x) \geq 1$ for any $x \in X$.
- A Q-monotone, Q-deflating mapping is called a *Q-order preconucleus*.
- An idempotent Q-order preconucleus is called a *Q-order conucleus*.

Proposition

Let (f, g) be a Q-adjunction between (X, e_X) and (Y, e_Y). Then $g \circ f$ is a Q-order nucleus on X, and $f \circ g$ is a Q-order conucleus on Y.

As any Q-sup-lattice homomorphism is a left Q-adjoint, we instantly get the following:

Proposition

Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and let $f : X \to Y$ be a homomorphism. Then $f^* \circ f$ is a Q-order nucleus on X, and $f \circ f^*$ is a Q-order conucleus on Y.

Definition

Let $(X, \bigsqcup)$ be a Q-sup-lattice, $Y \subseteq X$, and $M \in Q^Y$. We define an extension $M' \in Q^X$ of M as

$$M'(x) = \begin{cases} M(x) & x \in Y, \\ \perp & x \notin Y. \end{cases}$$

We say that Y is a *sub-Q-sup-lattice* of X if it is closed under Q-joins in the following sense: for any $M \in Q^Y$, the element $\bigsqcup M' \in X$ also belongs to Y.

Proposition

Let $(X, \bigsqcup)$ be a Q-sup-lattice, and j be a Q-order-prenucleus on X. Then the subset of fixed points of j, $X_j = \{x \in X \mid j(x) = x\}$ is a Q-sup-lattice with $\bigsqcup_{X_j} M = j(\bigsqcup_X M')$.

(X_j is closed under Q-meets, Q-joins are calculated using j)

Theorem

Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and $f\colon X \to Y$ be a surjective homomorphism. Then there exists a Q-order nucleus $j\colon X \to X$ such that $Y \cong X_j$.

Proposition

Let $(X, \bigsqcup)$ be a Q-sup-lattice, and $g\colon X \to X$ be a Q-order preconucleus on X. Then $X_g = \{x \in X \mid g(x) = x\}$ is a sub-Q-sup-lattice of X.

(Q-joins are inherited, Q-meets computed using g)

Theorem

Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and $f\colon X \to Y$ be an injective homomorphism. Then there exists a Q-order conucleus $g\colon Y \to Y$ such that $X \cong Y_g$.

S. A. Solovyov 2016: For a given commutative unital quantale Q, the categories Q-**Sup** and Q-**Mod** are isomorphic.

$F: Q\text{-}\mathbf{Mod} \to Q\text{-}\mathbf{Sup}$

1. $e_A(a_1, a_2) = a_1 \to_Q a_2$
2. $\bigsqcup M = \bigvee_{a \in A}(M(a) * a)$

$F: Q\text{-}\mathbf{Sup} \to Q\text{-}\mathbf{Mod}$

1. $a_1 \leq a_2$ iff $1 \leq e_A(a_1, a_2)$ for every $a_1, a_2 \in A$
2. $\bigvee S = \bigsqcup M_S^1$ for every $S \subseteq A$
3. $q * a = \bigsqcup M_a^q$ for every $q \in Q$ and every $a \in A$

where

$$M_S^q(x) = \begin{cases} q, & x \in S, \\ \bot & \text{otherwise.} \end{cases}$$

Existing results on Q-modules can be directly transferred to Q-**Sup**:

Proposition

*In the category Q-**Sup**, the following hold:*

1. *All monomorphisms are injective mappings.*
2. *All epimorphisms are surjective mappings.*
3. *All monomorphisms are regular, i.e., equalizers of some pair of homomorphisms.*
4. *All epimorphisms are regular, i.e., coequalizers of some pair of homomorphisms.*

Theorem

There exists the free Q-sup-lattice FX over any set X as $FX \cong Q^X$ with $e\left((q_x)_{x\in X}, (r_x)_{x\in X}\right) = (q_x) \rightarrow_Q (r_x)$.

Proposition

*The category Q-***Sup** *has tensor products.*

Proposition

*The category Q-***Sup** *with the tensor product and Q as the unit object is a symmetric closed monoidal category.*

Proposition

A Q-sup-lattice $(X, \bigsqcup)$ *is injective iff its dual* $(X, \bigsqcap)$ *is projective.*

Proposition

A Q-sup-lattice $(X, \bigsqcup)$ *is flat iff it is projective.*

Characterization of projectivity not yet available, just partial results such as:
A is projective if $A \cong \prod_{i \in I}(Q \cdot d_i)$ for some I where $d_i \cdot d_i = d_i$ for every $i \in I$.

Such isomorphism was shown between the categories Q-**Alg** and Q-**Quant** (the conversion behaves nicely wrt. to multiplication).

Q-sup-algebras – general algebras in the category Q-**Sup**

A *type* is a set Ω of *function symbols*.
To each $\omega \in \Omega$, a number $n \in \mathbb{N}_0$ is assigned (arity).
For each $n \in \mathbb{N}_0$, $\Omega_n \subseteq \Omega$ is the subset of all n-ary function symbols from Ω.

Definition

A *Q-ordered algebra*) is a triple $\mathcal{A} = (A, e, \Omega)$ where (A, e) is a Q-ordered set, (A, Ω) is an Ω-algebra, and each operation ω of non-zero arity is Q-monotone in any component, that is,

$$e(b, c) \leq e(\omega(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n), \omega(a_1, \ldots, a_{j-1}, c, a_{j+1}, \ldots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, and $a_1, \ldots, a_n, b, c \in A$.

Definition

Let (A, e_A, Ω) and (B, e_B, Ω) be Q-ordered algebras, and $\varphi \colon A \to B$ be a Q-monotone mapping. Then φ is called a *Q-ordered algebra subhomomorphism* if

$$1 \leq e_B(\omega_B(\varphi(a_1), \ldots, \varphi(a_n)), \varphi(\omega_A(a_1, \ldots, a_n)))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and

$$1 \leq e_B(\omega_B, \varphi(\omega_A))$$

for every $\omega \in \Omega_0$.

Definition

A *Q-sup-algebra of type* Ω (shortly, a *Q-sup-algebra*) is a triple $\mathcal{A} = (A, \bigsqcup, \Omega)$ where $(A, \bigsqcup)$ is a Q-sup-lattice, (A, Ω) is an Ω-algebra, and each operation ω is Q-join-preserving in any component, that is,

$$\omega\left(a_1, \ldots, a_{j-1}, \bigsqcup M, a_{j+1}, \ldots, a_n\right) = \bigsqcup \omega(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n)^{\rightarrow}_{Q}(M)$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, $a_1, \ldots, a_n \in A$, and $M \in Q^A$.

Definition

Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup algebras, and $\varphi\colon A \to B$ be a Q-join-preserving mapping. Then φ is called a *Q-sup-algebra homomorphism* if

$$\omega_B(\varphi(a_1), \dots, \varphi(a_n)) = \varphi(\omega_A(a_1, \dots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega$, and $a_1, \dots, a_n \in A$, and

$$\omega_B = \varphi(\omega_A)$$

for any $\omega \in \Omega_0$.

Category Q-**Sup**-Ω-**Alg**

As any Q-sup-algebra can be regarded as a Q-ordered algebra, the concept of subhomomorphism applies here as well.

Proposition

- *If $f: A \to B$ is a homomophism, then $f^*: B \to A$ is a subhomomorphism.*
- *Composition of subhomomorphisms is a subhomomorphism.*

Definition

Let $(A, \bigsqcup, \Omega)$ be a Q-sup-algebra.

- A Q-monotone, Q-inflating mapping which is a subhomomorphism is called a *prenucleus* on A.
- An idempotent prenucleus is called a *nucleus*.

Proposition

Let $\mathcal{A} = (A, \bigsqcup_A, \Omega)$ be a Q-sup-algebra, and $j\colon A \to A$ be a nucleus on $\mathcal{A}$. Put

1. $A_j = \{a \in A \mid j(a) = a\}$,
2. $\bigsqcup_{A_j} M = j\left(\bigsqcup_A M'\right)$ *for every* $M \in Q^{A_j}$,
3. $\omega_{A_j}(a_1, \ldots, a_n) = j(\omega_A(a_1, \ldots, a_n))$ *for every* $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A_j$, *and*
4. $\omega_{A_j} = j(\omega_A)$ *for any* $\omega \in \Omega_0$.

Then $(A_j, \bigsqcup_{A_j}, \Omega)$ is a Q-sup-algebra, and j is a Q-sup-algebra homomorphism from A to A_j.

Theorem

Let $f\colon A \to B$ be a surjective homomorphism of Q-sup-algebras. Then there exists a nucleus j on A such that $B \cong A_j$.

Definition

A Q-sup-algebra $(B, \bigsqcup_B, \Omega)$ is a *sub-Q-sup-algebra* of a Q-sup-algebra $(A, \bigsqcup_A, \Omega)$ if $B \subseteq A$, and the inclusion mapping is a Q-sup-algebra homomorphism.

Proposition

Let $g\colon A \to A$ be a conucleus on a Q-sup-algebra $\mathcal{A} = (A, \bigsqcup_A, \Omega)$. Then $(A_g, \bigsqcup_A|_{A_g}, \Omega)$ where $A_g = \{a \in A \mid g(a) = a\}$ is a sub-Q-sup-algebra of $\mathcal{A}$.

Theorem

Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup-algebras, and $f : A \to B$ be an injective homomorphism. Then there exists a conucleus $g\colon B \to B$ such that $A \cong B_g$.

Theorem

*The categories Q-**Mod**-Ω-**Alg** (Ω-algebras in the category of Q-modules) and Q-**Sup**-Ω-**Alg** are isomorphic.*

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Thank you for your attention.