Algebraic characterization of temporal logics on forests

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Roadmap

- Forests, trees
- Forest automata, regular forests
- The Forest Logics $\mathrm{FL}(\mathcal{L})$
- Algebraic characterization of the languages definable in the logics
- Application: a decidable fragment of CTL

Origins

The Forest Logics studied here is originating from the work of Ésik (2005) (difference: forests, separating tree and forest formulas) and similar to the work of Bojanczyk, Straubing and Walukiewicz (2012) (difference: modality evaluation is consistent for trees here). The general algebraic characterization and the decidability result of the given fragment draws heavily from Ésik (2005) and Ésik and Iván (2008).

Forests

An alphabet is a finite nonempty set A.

Trees, forests

Trees and forests over some alphabet A are defined via mutual recursion as

- if $t_1, t_2, \ldots, t_n$ are trees for some $n \geq 0$, then the ordered sum $t_1 + \ldots + t_n$ is a forest;
- if s is a forest and $a \in A$ is a symbol, then $a(s)$ is a tree.
- nothing else is a tree nor a forest.

- In particular, the empty forest, denoted $\mathbf{0}$ is a forest.
- For $A = \{a, b\}$, $a(\mathbf{0})$ is a tree and $a(\mathbf{0}) + b(\mathbf{0}) + a(\mathbf{0})$ is a forest. Omitting zeros: $a + b + a$.
- Also: $a(a + b(a)) + b(b + a(b)) + a$.

Automata

Forest automata

A (finite) forest automaton over an alphabet A is a system $\mathcal{M} = (Q, +, 0, A, \cdot)$ where

- $(Q, +, 0)$ is a (finite) monoid, its elements are also called states,
- the function $\cdot$ maps $A \times Q$ to Q.

Evaluation

In $\mathcal{M}$ above, each tree t and forest s evaluates to a state $t^{\mathcal{M}}$ and $s^{\mathcal{M}}$ respectively as

- $(t_1 + t_2 + \ldots + t_n)^{\mathcal{M}} = t_1^{\mathcal{M}} + t_2^{\mathcal{M}} + \ldots + t_n^{\mathcal{M}}$ and
- $(a(s))^{\mathcal{M}} = a \cdot (s^{\mathcal{M}})$.

In particular, the empty forest $\mathbf{0}$ evaluates to 0.

Forest language recognizers

A forest language (over A) is an arbitrary set of A-forests.

The forest automaton $\mathcal{M} = (Q, +, 0, A, \cdot)$ equipped with a set $F \subseteq Q$ of final states recognizes the forest language

$$L(\mathcal{M}, F) = \{s \in F_A : s^{\mathcal{M}} \in F\}.$$

A forest language is called regular if it can be recognized by some finite forest automaton by some set of final states.

Each regular forest language L has a minimal forest automaton $\mathcal{M}(L)$, unique up to isomorphism, which can be computed in polynomial time from any forest automaton recognizing L.

Logics

Forest formulas

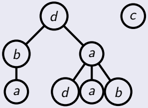
- $s \models \top$, $s \not\models \bot$
- $\varphi \vee \psi$, $\neg\varphi$
- $L \subseteq F_B$ a forest language over B, $(\varphi_b)_{b\in B}$ a set of tree formulas over A: $L[b \mapsto \varphi_b]_{b \in B}$

Tree formulas

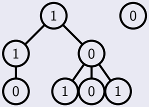
- all forest formulas, same semantics (a tree is a forest of a single tree)
- $a(s) \models a$
- $\varphi \vee \psi$, $\neg\varphi$

Example

$L_{\mathrm{EX}} = \{s \in F_{\{0,1\}} : \exists \text{ depth-one node labeled } 1\}$



$\varphi_0 = a \vee c$
$\varphi_1 = b \vee d$



$\in L_{\mathrm{EX}} \Rightarrow \models L_{\mathrm{EX}}(\varphi_0, \varphi_1)$

A class of logics

When $\mathcal{L}$ is a class of forest languages, let $\mathrm{FL}(\mathcal{L})$ stand for the logics whose forest formulas of the form $L[b \mapsto \varphi_b]$ are allowed only if $L \in \mathcal{L}$. Let $\mathbf{FL}(\mathcal{L})$ stand for the forest languages definable in $\mathrm{FL}(\mathcal{L})$.

Example

The forest language over $\{a, b, c\}$ consisting of exactly those forests having a depth-one node labeled a having a child labeled b having a child labeled a and a child labeled c is in $\mathbf{FL}(\{L_{\mathrm{EX}}\})$.

Definability problem

Fix some class $\mathcal{L}$ of regular forest languages (or "modalities").
Is it decidable, given some forest language L, with (say) its minimal forest automaton, whether $L \in \mathbf{FL}(\mathcal{L})$ holds?

Algebraic machinery: the Moore product

Let $\mathcal{M}_1 = (Q_1, +_1, 0_1, A, \cdot_1)$ and $\mathcal{M}_2 = (Q_2, +_2, 0_2, B, \cdot_2)$ be forest automata and $\alpha : Q_1 \times A \to B$ be a control function, then the Moore product $\mathcal{M}_1 \times_\alpha \mathcal{M}_2$ is the forest automaton

$$(Q_1 \times Q_2, +, (0_1, 0_2), A, \cdot)$$

with $(q_1, q_2) + (q_1', q_2') = (q_1 +_1 q_1', q_2 +_2 q_2')$ and

$$a \cdot (q_1, q_2) = (a \cdot_1 q_1, \alpha(a \cdot_1 q_1, a) \cdot_2 q_2).$$

A general theorem

Theorem

Suppose $\mathcal{L}$ is a class of regular forest languages such that for each modality $L \in \mathcal{L}$ and forest language K recognizable in $\mathcal{M}(L)$ we have $K \in \mathbf{FL}(\mathcal{L})$. Then the following are equivalent for any forest language L:

- L is definable in $\mathrm{FL}(\mathcal{L})$
- $\mathcal{M}(L)$ is contained in the least class of forest automata which contains each forest automaton $\mathcal{M}(K)$, $K \in \mathcal{L}$, and is closed under renamings, subautomata, homomorphic images and Moore products.

Note

This does not gives an effective characterization.

A specific logic

EF^*

The formulas of the logic EF^* over some alphabet A are

- $\top$, $\bot$ are forest formulas,
- $a \in A$ are tree formulas,
- $\varphi \vee \psi$, $\neg\varphi$ are also forest (tree) formulas if so are φ and ψ,
- every tree formula is a forest formula as well,
- $\mathrm{EF}^*\varphi$, with φ being a tree formula.

A forest s satisfies $\mathrm{EF}^*\varphi$ iff some subtree of s satisfies φ.

In particular, if $s = a(s')$ is a tree satisfying φ, then s satisfies $\mathrm{EF}^*\varphi$ as well.

An effective characterization

Theorem

Let $L \subseteq F_A$ be a regular forest language, $\mathcal{M}(L) = (Q, +, 0, A, \cdot)$ be its minimal forest automaton and $\leq$ be the reflexive-transitive closure of $q \leq a \cdot q$, $q \leq p + q$.
Then L is definable in EF^* if and only if all the following conditions hold:

- $(Q, +, 0)$ is a commutative monoid;
- $aap = ap$ for each $a \in A$, $p \in Q$;
- $\leq$ is a partial ordering;
- $p \leq q$ implies $p + q = q$ for each $p, q \in Q$.

Summary

- A class of logics is defined on forests (here: ordered vectors of finite, ordered, unranked trees).
- We gave an algebraic characterization of the languages definable in these logics by means of the Moore product.
- Using the above characterization we showed that definability in the CTL-fragment EF^* is decidable in low polynomial time.
- Many open problems remain, one of them being the definability of CTL itself.