On congruence extension and Hamiltonian ordered algebras

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Outline

1. Introduction
   - Background
2. Preliminaries
   - Ordered universal algebras
   - Congruence extension properties
3. Hamiltonian ordered algebras
   - Hamiltonian ordered algebras and CEP
   - Hamiltonian squares and congruence extension
4. Final remarks
   - Open problems

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The talk

This talk will cover some of my joint work [3] with Valdis Laan and Nasir Sohail on a property of partially ordered universal algebras and congruences thereof.

Namely, we studied when congruences on subalgebras of an algebra can be extended to the algebra itself.

The generic name for such a property is the congruence extension property (CEP) and the topic has seen intermittent study since the early seventies, notably in the cases of (unordered) semigroups and universal algebras.

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A primer on ordered universal algebras...

Definition 1

Let Ω be a type. An ordered Ω-algebra (or simply ordered algebra) is a triple $\mathcal{A} = (A, \Omega_A, \leq_A)$ comprising a poset $(A, \leq_A)$ and a set Ω_A of monotone operations on A, where $\omega_A \in \Omega_A$ has arity k if $\omega \in \Omega_k$ and where the monotonicity of ω_A means that

$$a_1 \leq_A a_1' \wedge \ldots \wedge a_k \leq_A a_k' \Longrightarrow \omega_A(a_1, \ldots, a_k) \leq_A \omega_A(a_1', \ldots, a_k')$$

for all $a_1, \ldots, a_k, a_1', \ldots, a_k' \in A$.

Definition 2

An ordered algebra $\mathcal{B} = (B, \Omega_B, \leq_B)$ is called a subalgebra of $\mathcal{A} = (A, \Omega_A, \leq_A)$ if

1. $B \subseteq A$,
2. for every k and for every $\omega \in \Omega_k$, $\omega_B = \omega_A|_{B^k}$,
3. $\leq_B = \leq_A \cap (B \times B)$.

The order and operations on the direct product of ordered algebras are defined componentwise.

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The order and operations on the direct product of ordered algebras are defined componentwise.

A homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of ordered algebras is a monotone operation-preserving mapping from an ordered Ω-algebra $\mathcal{A}$ to an ordered Ω-algebra $\mathcal{B}$.

A homomorphism is called f an order-embedding if additionally

$$f(a) \leq_B f(a') \Longrightarrow a \leq_A a'$$

for all $a, a' \in A$.

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for all $a, a' \in A$.

Given a quasiorder θ on a poset $(A, \leq)$ and $a, a' \in A$, we write

$$a \underset{\theta}{\leq} a' \iff (\exists n \in \mathbb{N})(\exists a_1, \ldots, a_n \in A)(a \leq a_1 \,\theta\, a_2 \leq a_3 \,\theta\, \ldots \,\theta\, a_n \leq a').$$

Definition 3

An order-congruence on an ordered algebra $\mathcal{A}$ is an algebraic congruence θ such that the closed chains condition is satisfied:

$$(\forall a, a' \in A)\left(a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Longrightarrow a \,\theta\, a'\right).$$

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...and compatible quasiorders.

Definition 4

We call a preorder σ on an ordered algebra $\mathcal{A}$ a compatible quasiorder if it is compatible with operations and extends the order of $\mathcal{A}$, i.e. $\leq_A \subseteq \sigma$.

Such relations have also been called admissible preorders.

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Congruence extension properties

Definition 5

We say that an ordered algebra $\mathcal{A}$ has the congruence extension property (CEP) with respect to subalgebra $\mathcal{B}$ if every order-congruence θ on $\mathcal{B}$ is the restriction of some order-congruence Θ on $\mathcal{A}$, i.e. $\Theta \cap (B \times B) = \theta$.

Definition 6

We say that an ordered algebra $\mathcal{A}$ has the quasiorder extension property (QEP) with respect to subalgebra $\mathcal{B}$ if every compatible quasiorder σ on $\mathcal{B}$ is the restriction of some compatible quasiorder Σ on $\mathcal{A}$, i.e. $\Sigma \cap (B \times B) = \sigma$.

Proposition 7

If an ordered algebra $\mathcal{A}$ has QEP with respect to subalgebra $\mathcal{B}$ then it has CEP with respect to $\mathcal{B}$.

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We say that an ordered algebra $\mathcal{A}$ has the strong congruence extension property (SCEP) if any order-congruence θ on any subalgebra $\mathcal{B}$ of $\mathcal{A}$ can be extended to an order-congruence Θ on $\mathcal{A}$ in such a way that $\Theta \cap (B \times A) = \theta$.

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We say that an ordered algebra $\mathcal{A}$ has the strong quasiorder extension property (SQEP) if for any compatible quasiorder σ on any subalgebra $\mathcal{B}$ of $\mathcal{A}$ there exists a compatible quasiorder Σ on $\mathcal{A}$ such that

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Proposition 10

If an ordered algebra $\mathcal{A}$ has SQEP with respect to subalgebra $\mathcal{B}$ then it has SCEP and QEP with respect to $\mathcal{B}$. If $\mathcal{A}$ has SCEP with respect to $\mathcal{B}$ then it has CEP with respect to $\mathcal{B}$.

Proposition 11

If an ordered algebra $\mathcal{A}$ has SQEP or SCEP then all its subalgebras are convex.

- A subposet $\mathcal{B}$ of a poset $\mathcal{A}$ is called convex if $\forall b, b' \in \mathcal{B}\ \forall a \in \mathcal{A}\, (b \leq a \leq b' \implies a \in \mathcal{B})$.

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Hamiltonian algebras

Definition 12

An unordered algebra $\mathcal{A}$ is called Hamiltonian if every subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a class of some congruence on $\mathcal{A}$.

Observe that if θ is an order-congruence on an ordered algebra $\mathcal{A}$ then every θ-class is a convex subset of $\mathcal{A}$. Therefore it makes sense to distinguish between two kinds of ordered Hamiltonian algebras.

Definition 13

We say that an ordered algebra $\mathcal{A}$ is

1. Hamiltonian if every convex subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a class of some order-congruence on $\mathcal{A}$,
2. strongly Hamiltonian if every subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a class of some order-congruence on $\mathcal{A}$.

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Hamiltonian algebras and congruence extension

We have the following results relating Hamiltonian-ness and congruence extension properties:

Proposition 14

An ordered algebra is strongly Hamiltonian if and only if it is Hamiltonian and its subalgebras are convex.

Theorem 15

An ordered algebra has SCEP if and only if it is strongly Hamiltonian and has CEP.

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Hamiltonian squares and congruence extension

If an algebra has a Hamiltonian square, we have the SCEP with an extra assumption:

Theorem 17

Let $\mathcal{B}$ be a convex subalgebra of an ordered algebra $\mathcal{A}$, where $\mathcal{A} \times \mathcal{A}$ is Hamiltonian. If θ is an order-congruence on $\mathcal{B}$ which is a convex subset of $B \times B$ then there exists an order-congruence Θ on $\mathcal{A}$ such that $\Theta \cap (B \times A) = \theta$.

This assumption of convexity is necessary if we want to leverage the Hamiltonian-ness of $\mathcal{A} \times \mathcal{A}$, but it turns out to be rather strong:

Lemma 18

An order-congruence θ on an ordered algebra $\mathcal{A}$ is a convex subset of $\mathcal{A} \times \mathcal{A}$ if and only if θ extends the order of A (i.e. $\leq_A \subseteq \theta$).

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If we restrict ourselves to up-closed or down-closed subalgebras, then we also get the SQEP:

Theorem 19

Let $\mathcal{B}$ be an up-closed (or down-closed) subalgebra of an ordered algebra $\mathcal{A}$, where $\mathcal{A} \times \mathcal{A}$ is Hamiltonian. If σ is a compatible quasiorder on $\mathcal{B}$ which is a convex subset of $B \times B$ then then there exists a compatible quasiorder Σ on $\mathcal{A}$ such that $\Sigma \cap (B \times B) = \sigma$ and $\Sigma \cap \Sigma^{-1} \cap (B \times A) = \sigma \cap \sigma^{-1}$.

The SCEP is also transferable from the underlying unordered algebra in this case:

Proposition 20

Let $\mathcal{B}$ be an up-closed (or down-closed) subalgebra of an ordered algebra $\mathcal{A}$. Assume that $\mathcal{A}$ has the algebraic SCEP with respect to $\mathcal{B}$. Then $\mathcal{A}$ has SCEP with respect to $\mathcal{B}$ as an ordered algebra.

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Origins...

Theorem 17 is patterned after the following result of V. Gould and M. Wild:

Theorem 21

If $\mathcal{A}$ is an algebra such that $\mathcal{A} \times \mathcal{A}$ is Hamiltonian, then $\mathcal{A}$ has the SCEP. In particular, each Hamiltonian variety of unordered algebras has the SCEP.

This raises the question whether

Problem 22

Each Hamiltonian variety of ordered algebras has the (ordered) (S)CEP.

Here, variety means an inequational class, i.e. a class of all ordered algebras satisfying some set of inequalities.

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Each strongly Hamiltonian variety of ordered algebras has the SCEP.

One could also ask for more, namely whether the convexity assumptions in Theorem 17 are necessary and perhaps one could prove this theorem without them, or under weaker assumptions, like all subalgebras of $\mathcal{A}$ needing to be Hamiltonian.

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Well, at least one in the hand...

The End

Thank you for listening!

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