Semiring Structures Based on Meet and Plus Ideals in Lower BCK-semilattices

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In this presentation, I introduce the notion of the meet set based on two subsets of a lower BCK-semilattice X and I discuss conditions for the meet set to be a (positive implicative, commutative, implicative) ideal. Also introduced the meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X. I induce the semiring structure by using meet operation and addition.

BCI-ALGEBRA

Definition

An algebra $(X; *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions: for any x, y, z $\in X$,
BCI-1: $((x * y) * (x * z)) * (z * y) = 0$
BCI-2: $x * 0 = x$
BCI-3: $x * y = 0$ and $y * x = 0$ imply $x = y$

Proposition

Suppose that X is a BCI-algebra. Define a binary relation $\leq$ on X by which $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. Then $(X; \leq)$ is a partially ordered set with 0 as a minimal element.

BCK-ALGEBRA

Definition

Given a BCI-algebra X, if it satisfies the condition
BCK-1: $0 * x = 0$ for all $x \in X$, which means that $0 \leq x$. for each $x \in X$.
We call this algebra a BCK-algebra.

Definition

A partially ordered set $(X; \leq)$ is called a lower semilattice if any two elements of X have the greatest lower bound. It is called an upper semilattice if each pair of elements in X has its least upper bound.
Given a BCK-algebra X, if it with respect to its BCI-ordering $\leq$ forms a lower semilattice, then the algebra X is called a lower BCK-semilattice. Similarly we can define an upper BCK-semilattice.
In a lower BCK-semilattice we denote $x \wedge y = glb\{x, y\}$.

Ideal

Definition

A subset A of a BCI-algebra X is called an ideal of X if
(i) $0 \in A$.
(ii) $x \in A$ and $y * x \in A$ imply $y \in A$ for any $x, y \in X$.
Note that X and $\{0\}$ are ideals of X, and they are called the trivial ideals of X.

Definition

Let S be a subset of a BCI-algebra X. We call the least ideal of X, containing S, the generated ideal of X by S, denoted by $< S >$ or $(S]$.

Meet Ideal

In what follows, let X be a lower BCK-semilattice unless otherwise specified.
For any nonempty subsets A and B of X, we consider the set

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

where $a \wedge b$ is the greatest lower bound of a and b. We say that K is the *meet set* based on A and B.

Note that $A \cap B \subseteq K$, but the reverse inclusion is not true as seen in the following example.

Example

Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\} \not\subseteq A \cap B.$$

The following example shows that the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ may not be an ideal of X for some subsets A and B of X.

Example

Let $X = \{0, 1, 2, 3, 4\}$ be a lower BCK-semilattice with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then X has four ideals $A_0 = \{0\}$, $A_1 = \{0,1,3\}$, $A_2 = \{0,2,4\}$ and $A_4 = X$. For subsets $A = \{2,3\}$ and $B = \{1,4\}$ of X, we have $2 \wedge 1 = 0$, $2 \wedge 4 = 2$, $3 \wedge 1 = 1$ and $3 \wedge 4 = 0$. Thus

$$\{a \wedge b \mid a \in A,\ b \in B\} = \{0,1,2\},$$

which is not an ideal of X.

Definition

For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation*.

If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Obviously, $A \wedge B = B \wedge A$ for any nonempty subsets A and B of X.

Theorem
For any nonempty subsets A, B and C of X, we have

$$A \subseteq B,\ A \subseteq C \ \Rightarrow\ A \subseteq B \wedge C.$$

The following example shows that there are subsets A, B and C of X such that $A \subseteq B$ and $A \subseteq C$, but $B \wedge C \nsubseteq A$.

Example

Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of X, we have

$$B \wedge C = \langle \{a \wedge b \mid a \in A,\ b \in B\} \rangle = \{0, 1, 2\} \nsubseteq \{0, 1\} = A.$$

We provide conditions for the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B to be an ideal.

Theorem

If A and B are ideals of X, then so is the meet set

$$A \wedge B := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Definition

An ideal A of a BCK-algebra X is called commutative if $(x*y)*z \in A$ and $z \in A$ implies $x*(y*(y*x)) \in A$ for any $x, y \in X$.

Definition

An ideal A of a BCK-algebra X is called implicative if $(x*(y*x))*z \in A$ and $z \in A$ implies $x \in A$, for any $x, y, z \in X$.

Definition

An ideal A of a BCK-algebra X is called positive implicative if $(x*y)*z \in A$ and $y*z \in A$ implies $x*z \in A$, for any $x, y, z \in X$.

Lemma

For an ideal A of a BCK-algebra X, the following are equivalent.

(i) *A is positive implicative.*

(ii) $(\forall x, y \in X)\,((x * y) * y \in A \;\Rightarrow\; x * y \in A)$.

Lemma

For an ideal A of a BCK-algebra X, the following are equivalent.

(i) *A is commutative.*

(ii) $(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; x * (y * (y * x)) \in A)$.

Lemma

Let A be an ideal of a BCK-algebra X. Then A is implicative if and only if A is both positive implicative and commutative.

Theorem

If A and B are positive implicative (resp., commutative, implicative) ideals of X, then so is the meet set

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Proposition

If A, B and C are ideals of X, then

$$A \wedge \{0\} = \{0\}. \tag{1}$$

$$A \wedge B = A \cap B. \tag{2}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C) = \{a \wedge b \wedge c \mid a \in A, b \in B, c \in C\}. \tag{3}$$

Corollary

For ideals $A_1, A_2, \cdots, A_n$ of X

$$\bigwedge_{i=1}^{n} A_i := A_1 \wedge A_2 \wedge \cdots \wedge A_n$$
$$= \{a_1 \wedge a_2 \wedge \cdots \wedge a_n \mid a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\} \quad (4)$$
$$= \bigcap_{i=1}^{n} A_i.$$

Theorem

For any element a of X, we have

$$\langle a \rangle = a \wedge X. \tag{5}$$

RELATIVE ANNIHILATORS

Definition

For any nonempty subsets A and B of X, we define a set

$$(A :_\wedge B) := \{x \in X \mid x \wedge B \subseteq A\} \tag{6}$$

whenever $x \wedge B$ exists for all $x \in X$, and it is called the *relative annihilator* of B with respect to A.

For a nonempty subset B of a lower BCK-semilattice X, consider the following condition:

$$(\forall x, y \in X)(\forall b \in B)\left((x \wedge b) * (y \wedge b) \leq (x * y) \wedge b\right). \tag{7}$$

We provide conditions for the relative annihilator of a set with respect to a set to be an ideal.

Theorem

Let B be a nonempty subset of a lower BCK-semilattice X in which the condition (7) *is valid. If A is an ideal of X, then the relative annihilator* $(A :_\wedge B)$ *of B with respect to A is an ideal of X.*

Theorem

If A and B are ideals of a lower BCK-semilattice X, then the relative annihilator $(A :_{\wedge} B)$ of B with respect to A is an ideal of X.

Theorem

For ideals A and B of a lower BCK-semilattice X, if A is positive implicative (resp., commutative and implicative), then the relative annihilator $(A :_\wedge B)$ of B with respect to A is a positive implicative (resp., commutative and implicative) ideal of X.

Definition

A mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq c(A))\,, \tag{8}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \ \Rightarrow\ c(A) \subseteq c(B))\,. \tag{9}$$

If a weak closure operation $c : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(c(c(A)) = c(A))\,, \tag{10}$$

then we say that c is a closure operation on $\mathcal{I}(X)$. In what follows, we use A^{cl} instead of $c(A)$.

Definition

An element x of X is called a *zeromeet element* of X if the condition

$$(\exists y \in X \setminus \{0\})\,(x \wedge y = 0)$$

is valid. Otherwise, x is called a *non-zeromeet element* of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Definition

A weak closure operation "*cl*" on $\mathcal{I}(X)$ is said to be *meet* if it satisfies:

$$(\forall A \in \mathcal{I}(X))\,(\forall a \in X \setminus Z(X))\,\left((a \wedge A)^{cl} = a \wedge A^{cl}\right). \qquad (11)$$

We consider relations between $(z \wedge A :_\wedge z)$ and A for any ideal A and $z \in X \setminus Z(X)$. We can easily prove that $A \subseteq (z \wedge A :_\wedge z)$. But the reverse inclusion is not true in general as seen in the following example.

Example

Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For a non-zeromeet element 2 and an ideal $A = \{0, 1, 2\}$ of X, we have

$$(2 \wedge \{0, 1, 2\} :_\wedge 2) = X \not\subseteq A.$$

In the following proposition, we discuss conditions for the inclusion $(z \wedge A :_{\wedge} z) \subseteq A$ to be true. We first consider the following condition:

$$(\forall a, b \in X)(\forall z \in X \setminus Z(X))((a * b) \wedge z \leq (a \wedge z) * (b \wedge z)), \quad (12)$$

Theorem

If X satisfies the condition (12), then $(z \wedge A :_{\wedge} z) \subseteq A$ and hence $(z \wedge A :_{\wedge} z) = A$ for every $A \in \mathcal{I}(X)$ and $z \in X \setminus Z(X)$.

Theorem

Let X satisfy the condition (12) *and let "cl" be a weak closure operation on* $\mathcal{I}(X)$. *Then "cl" is meet if and only if it satisfies the following properties:*

$$\langle a\rangle^{cl} = \langle a\rangle \text{ and } A^{cl} = ((a \wedge A)^{cl} :_{\wedge} a) \tag{13}$$

for any $a \in X \setminus Z(X)$ *and any ideal A of X.*

Plus Ideal

Definition

For any nonempty subsets A and B of X, denote by $A + B$ the ideal generated by $A \cup B$, and is called the *plus ideal* of A and B. Thus

$$A + B = \langle A \cup B \rangle.$$

The operation "+" is called the *addition*. Obviously,

1. $A, B \subseteq A + B$,
2. $A + \{0\} = A$
3. $A + B = B + A$.

Example

Consider a lower *BCK*-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Them X has ten ideals $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 3\}$, $A_3 = \{0, 1, 3\}$, $A_4 = \{0, 1, 2\}$, $A_5 = \{0, 1, 4\}$, $A_6 = \{0, 1, 2, 3\}$, $A_7 = \{0, 1, 3, 4\}$, $A_8 = \{0, 1, 2, 4\}$ and $A_9 = X$. For subsets $A = \{1, 3\}$ and $B = \{2\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\} = A_6.$$

Given two nonempty subsets A and B of X, we note that every ideal I of X is represented by the meet ideal based on some A and B, and every ideal J of X is represented by the plus ideal of A and B. But we know that they are different, that is, $I \neq J$ in general as seen in the following example.

Example

Consider a lower *BCK*-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For two subsets $A = \{1\}$ and $B = \{2, 3\}$ of X, the ideal $I = \{0, 1\}$ is represented by the meet ideal based on A and B as follows

$$I = \langle A \wedge B \rangle = \langle \{0, 1\} \rangle = \{0, 1\}.$$

Also the ideal $J = \{0, 1, 2, 3\}$ is represented by the plus ideal of A and B as follows:

$$J = A + B = \langle A \cup B \rangle = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

We know that $I \neq J$.
But we have the following Theorem;

Theorem

For any nonempty subsets A and B of X, we have $A \wedge B \subseteq A + B$.

For any nonempty subsets A, B and C of X, consider the following condition.

$$A \subseteq C,\ B \subseteq C\ \Rightarrow A + B \subseteq C. \tag{14}$$

The condition (14) is not valid in general as we can see in the following example.

Example

Consider a lower *BCK*-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\} \nsubseteq C.$$

We provide conditions for the implication (14) to be hold.

Theorem

If A and B are nonempty subsets of X and C is an ideal of X, then the implication

$$A \subseteq C,\ B \subseteq C\ \Rightarrow A + B \subseteq C$$

is valid.

Theorem

For any ideals A, B and C of a commutative BCK-algebra X, we have

$$A \wedge (B + C) = (A \wedge B) + (A \wedge C)$$

and

$$(B + C) \wedge A = (B \wedge A) + (C \wedge A).$$

SEMIRING

Theorem

Let $\mathcal{I}(X)$ be the set of all ideals of a commutative BCK-algebra X. Then $(\mathcal{I}(X), +, \wedge)$ is a semiring, that is, two operations $+$ and $\wedge$ are associative on $\mathcal{I}(X)$ such that

(i) *addition $+$ is a commutative operation,*

(ii) *there exist $\{0\} \in \mathcal{I}(X)$ such that $A + \{0\} = A$ and $A \wedge \{0\} = \{0\} \wedge A = \{0\}$ for each $A \in \mathcal{I}(X)$, and*

(iii) *the meet operation $\wedge$ distributes over addition $(+)$ both from the left and from the right.*

Thanks For Your Attention

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